MATHEMATICAL TRIPOS Part II Alternative A

Wednesday 6 June 2001 9.00 to 12.00

PAPER 3

Before you begin read these instructions carefully.

Each question is divided into Part ***(i)*** *and Part* ***(ii)****, which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* ***SIX*** *questions.*

The number of marks for each question is the same, with Part ***(ii)*** *of each question carrying twice as many marks as Part* ***(i)****. Additional credit will be given for a substantially complete answer to either part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, . . . , L*** *according to the letter affixed to each question. (For example,* ***10E, 12E*** *should be in one bundle and* ***1D, 11D*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing **all** *Parts of* **all** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

(i) Explain what is meant by the *transition semigroup* $\{P_t\}$ of a Markov chain X in continuous time. If the state space is finite, show under assumptions to be stated clearly, that $P_t' = GP_t$ for some matrix G. Show that a distribution π satisfies $\pi G = 0$ if and only if $\pi P_t = \pi$ for all $t \geqslant 0$, and explain the importance of such π.

(ii) Let X be a continuous-time Markov chain on the state space $S = \{1, 2\}$ with generator

$$G = \begin{pmatrix} -\beta & \beta \\ \gamma & -\gamma \end{pmatrix}, \qquad \text{where} \quad \beta, \gamma > 0.$$

Show that the transition semigroup $P_t = \exp(tG)$ is given by

$$(\beta + \gamma)P_t = \begin{pmatrix} \gamma + \beta h(t) & \beta(1 - h(t)) \\ \gamma(1 - h(t)) & \beta + \gamma h(t) \end{pmatrix},$$

where $h(t) = e^{-t(\beta+\gamma)}$.

For $0 < \alpha < 1$, let

$$H(\alpha) = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{pmatrix}.$$

For a continuous-time chain X, let M be a matrix with (i, j) entry $P(X(1) = j \mid X(0) = i)$, for $i, j \in S$. Show that there is a chain X with $M = H(\alpha)$ if and only if $\alpha > \frac{1}{2}$.

2H Principles of Dynamics

(i) **(a)** Write down Hamilton's equations for a dynamical system. Under what condition is the Hamiltonian a constant of the motion? What is the condition for one of the momenta to be a constant of the motion?

(b) Explain the notion of an adiabatic invariant. Give an expression, in terms of Hamiltonian variables, for one such invariant.

(ii) A mass m is attached to one end of a straight spring with potential energy $\frac{1}{2}kr^2$, where k is a constant and r is the length. The other end is fixed at a point O. Neglecting gravity, consider a general motion of the mass in a plane containing O. Show that the Hamiltonian is given by

$$H = \frac{1}{2}\frac{p_\theta^2}{mr^2} + \frac{1}{2}\frac{p_r^2}{m} + \frac{1}{2}kr^2, \tag{1}$$

where θ is the angle made by the spring relative to a fixed direction, and p_θ, p_r are the generalised momenta. Show that p_θ and the energy E are constants of the motion, using Hamilton's equations.

If the mass moves in a non-circular orbit, and the spring constant k is slowly varied, the orbit gradually changes. Write down the appropriate adiabatic invariant $J\,(E, p_\theta, k, m)$. Show that J is proportional to

$$\sqrt{mk}\,(r_+ - r_-)^2\,,$$

where

$$r_\pm^2 = \frac{E}{k} \pm \sqrt{\left(\frac{E}{k}\right)^2 - \frac{p_\theta^2}{mk}}.$$

Consider an orbit for which p_θ is zero. Show that, as k is slowly varied, the energy $E \propto k^\alpha$, for a constant α which should be found.

[*You may assume without proof that*

$$\int_{r_-}^{r_+} dr \sqrt{\left(1 - \frac{r^2}{r_+^2}\right)\left(1 - \frac{r_-^2}{r^2}\right)} = \frac{\pi}{4r_+}\,(r_+ - r_-)^2\,.\Big]$$

3A Functional Analysis

(i) Define the notion of a measurable function between measurable spaces. Show that a continuous function $\mathbb{R}^2 \to \mathbb{R}$ is measurable with respect to the Borel σ-fields on $\mathbb{R}^2$ and $\mathbb{R}$.

By using this, or otherwise, show that, when $f, g : X \to \mathbb{R}$ are measurable with respect to some σ-field $\mathcal{F}$ on X and the Borel σ-field on $\mathbb{R}$, then $f + g$ is also measurable.

(ii) State the Monotone Convergence Theorem for $[0, \infty]$-valued functions. Prove the Dominated Convergence Theorem.

[*You may assume the Monotone Convergence Theorem but any other results about integration that you use will need to be stated carefully and proved.*]

Let X be the real Banach space of continuous real-valued functions on $[0, 1]$ with the uniform norm. Fix $u \in X$ and define

$$T : X \to \mathbb{R} \; ; \quad f \mapsto \int_0^1 f(t)u(t) \, dt \; .$$

Show that T is a bounded, linear map with norm

$$||T|| = \int_0^1 |u(t)| \, dt \; .$$

Is it true, for every choice of u, that there is function $f \in X$ with $||f|| = 1$ and $||T(f)|| = ||T||$?

4C Groups, Rings and Fields

(i) Let G be the cyclic subgroup of $GL_2(\mathbf{C})$ generated by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$, acting on the polynomial ring $\mathbf{C}[X, Y]$. Determine the ring of invariants $\mathbf{C}[X, Y]^G$.

(ii) Determine $\mathbf{C}[X, Y]^G$ when G is the cyclic group generated by $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

[*Hint: consider the eigenvectors.*]

5J Electromagnetism

(i) Develop the theory of electromagnetic waves starting from Maxwell equations in vacuum. You should relate the wave-speed c to ϵ_0 and μ_0 and establish the existence of plane, plane-polarized waves in which $\mathbf{E}$ takes the form

$$\mathbf{E} = (E_0 \cos(kz - \omega t), 0, 0) \ .$$

You should give the form of the magnetic field $\mathbf{B}$ in this case.

(ii) Starting from Maxwell's equation, establish Poynting's theorem.

$$-\mathbf{j} \cdot \mathbf{E} \ = \ \frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} \ ,$$

where $W = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2$ and $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \wedge \mathbf{B}$. Give physical interpretations of W, S and the theorem.

Compute the averages over space and time of W and $\mathbf{S}$ for the plane wave described in (i) and relate them. Comment on the result.

6K Dynamics of Differential Equations

(i) Define a hyperbolic fixed point x_0 of a flow determined by a differential equation $\dot{x} = f(x)$ where $x \in R^n$ and f is C^1 (i.e. differentiable). State the Hartman-Grobman Theorem for flow near a hyperbolic fixed point. For nonlinear flows in R^2 with a hyperbolic fixed point x_0, does the theorem necessarily allow us to distinguish, on the basis of the linearized flow near x_0 between (a) a stable focus and a stable node; and (b) a saddle and a stable node? Justify your answers briefly.

(ii) Show that the system:

$$\begin{aligned}
\dot{x} &= -(\mu + 1) + (\mu - 3)x - y + 6x^2 + 12xy + 5y^2 - 2x^3 - 6x^2y - 5xy^2, \\
\dot{y} &= 2 - 2x + (\mu - 5)y + 4xy + 6y^2 - 2x^2y - 6xy^2 - 5y^3
\end{aligned}$$

has a fixed point $(x_0, 0)$ on the x-axis. Show that there is a bifurcation at $\mu = 0$ and determine the stability of the fixed point for $\mu > 0$ and for $\mu < 0$.

Make a linear change of variables of the form $u = x - x_0 + \alpha y$, $v = x - x_0 + \beta y$, where α and β are constants to be determined, to bring the system into the form:

$$\begin{aligned}
\dot{u} &= v + u[\mu - (u^2 + v^2)] \\
\dot{v} &= -u + v[\mu - (u^2 + v^2)]
\end{aligned}$$

and hence determine whether the periodic orbit produced in the bifurcation is stable or unstable, and whether it exists in $\mu < 0$ or $\mu > 0$.

7C Geometry of Surfaces

(i) Give the definition of the surface area of a parametrized surface in $\mathbf{R}^3$ and show that it does not depend on the parametrization.

(ii) Let $\varphi(u) > 0$ be a differentiable function of u. Consider the surface of revolution:

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto f(u,v) = \begin{pmatrix} \varphi(u)\cos(v) \\ \varphi(u)\sin(v) \\ u \end{pmatrix}.$$

Find a formula for each of the following:

(a) The first fundamental form.

(b) The unit normal.

(c) The second fundamental form.

(d) The Gaussian curvature.

8B Logic, Computation and Set Theory

(i) Write down a set of axioms for the theory of dense linear order with a bottom element but no top element.

(ii) Prove that this theory has, up to isomorphism, precisely one countable model.

9C Number Theory

(i) State the law of quadratic reciprocity.

Prove that 5 is a quadratic residue modulo primes $p \equiv \pm 1 \pmod{10}$ and a quadratic non-residue modulo primes $p \equiv \pm 3 \pmod{10}$.

Determine whether 5 is a quadratic residue or non-residue modulo 119 and modulo 539.

(ii) Explain what is meant by the continued fraction of a real number θ. Define the convergents to θ and write down the recurrence relations satisfied by their numerators and denominators.

Use the continued fraction method to find two solutions in positive integers x, y of the equation $x^2 - 15y^2 = 1$.

10E Algorithms and Networks

(i) Let P be the problem

$$\text{minimize } \; f(x) \qquad \text{subject to } \; h(x) = b, \quad x \in X.$$

Explain carefully what it means for the problem P to be *Strong Lagrangian.*

Outline the main steps in a proof that a quadratic programming problem

$$\text{minimize } \; \frac{1}{2} x^T Q x + c^T x \qquad \text{subject to } \; Ax \geqslant b\,,$$

where Q is a symmetric positive semi-definite matrix, is Strong Lagrangian.

[*You should carefully state the results you need, but should not prove them.*]

(ii) Consider the quadratic programming problem:

$$\text{minimize } \; x_1^2 + 2x_1x_2 + 2x_2^2 + x_1 - 4x_2$$

$$\text{subject to } \; 3x_1 + 2x_2 \leqslant 6, \quad x_1 + x_2 \geqslant 1\,.$$

State necessary and sufficient conditions for (x_1, x_2) to be optimal, and use the active-set algorithm (explaining your steps briefly) to solve the problem starting with initial condition $(2, 0)$. Demonstrate that the solution you have found is optimal by showing that it satisfies the necessary and sufficient conditions stated previously.

11D Stochastic Financial Models

(i) Suppose that Z is a random variable having the normal distribution with $\mathbb{E}Z = \beta$ and Var $Z = \tau^2$.

For positive constants a, c show that

$$\mathbb{E}\left(ae^Z - c\right)_+ = ae^{\left(\beta+\tau^2/2\right)}\Phi\left(\frac{\log(a/c)+\beta}{\tau}+\tau\right) - c\Phi\left(\frac{\log(a/c)+\beta}{\tau}\right),$$

where Φ is the standard normal distribution function.

In the context of the Black-Scholes model, derive the formula for the price at time 0 of a European call option on the stock at strike price c and maturity time t_0 when the interest rate is ρ and the volatility of the stock is σ.

(ii) Let p denote the price of the call option in the Black-Scholes model specified in (i). Show that $\frac{\partial p}{\partial \rho} > 0$ and sketch carefully the dependence of p on the volatility σ (when the other parameters in the model are held fixed).

Now suppose that it is observed that the interest rate lies in the range $0 < \rho < \rho_0$ and when it changes it is linked to the volatility by the formula $\sigma = \ln(\rho_0/\rho)$. Consider a call option at strike price $c = S_0$, where S_0 is the stock price at time 0. There is a small increase $\Delta\rho$ in the interest rate; will the price of the option increase or decrease (assuming that the stock price is unaffected)? Justify your answer carefully.

[*You may assume that the function* $\phi(x)/\Phi(x)$ *is decreasing in* x, $-\infty < x < \infty$, *and increases to* $+\infty$ *as* $x \to -\infty$, *where* Φ *is the standard-normal distribution function and* $\phi = \Phi'$.]

12E Principles of Statistics

(i) Explain what is meant by a *uniformly most powerful unbiased test* of a null hypothesis against an alternative.

Let $Y_1, \ldots, Y_n$ be independent, identically distributed $N(\mu, \sigma^2)$ random variables, with σ^2 known. Explain how to construct a uniformly most powerful unbiased size α test of the null hypothesis that $\mu = 0$ against the alternative that $\mu \neq 0$.

(ii) Outline briefly the Bayesian approach to hypothesis testing based on *Bayes factors*.

Let the distribution of $Y_1, \ldots, Y_n$ be as in (i) above, and suppose we wish to test, as in (i), $\mu = 0$ against the alternative $\mu \neq 0$. Suppose we assume a $N(0, \tau^2)$ prior for μ under the alternative. Find the form of the Bayes factor B, and show that, for fixed n, $B \to \infty$ as $\tau \to \infty$.

13F Foundations of Quantum Mechanics

(i) Write the Hamiltonian for the harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

in terms of creation and annihilation operators, defined by

$$a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x - i\frac{p}{m\omega}\right), \qquad a = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}} \left(x + i\frac{p}{m\omega}\right).$$

Obtain an expression for $[a^\dagger, a]$ by using the usual commutation relation between p and x. Deduce the quantized energy levels for this system.

(ii) Define the number operator, N, in terms of creation and annihilation operators, $a^\dagger$ and a. The normalized eigenvector of N with eigenvalue n is $|n\rangle$. Show that $n \geq 0$.

Determine $a|n\rangle$ and $a^\dagger|n\rangle$ in the basis defined by $\{|n\rangle\}$.

Show that

$$a^{\dagger m} a^m |n\rangle = \begin{cases} \frac{n!}{(n-m)!}|n\rangle, & m \leq n\,, \\ 0\,, & m > n\,. \end{cases}$$

Verify the relation

$$|0\rangle\langle 0| = \sum_{m=0} \frac{1}{m!}(-1)^m a^{\dagger m} a^m\,,$$

by considering the action of both sides of the equation on an arbitrary basis vector.

14J Statistical Physics and Cosmology

(i) A spherically symmetric star has pressure $P(r)$ and mass density $\rho(r)$, where r is distance from the star's centre. Stating without proof any theorems you may need, show that mechanical equilibrium implies the Newtonian pressure support equation

$$P' = -\frac{Gm\rho}{r^2} ,$$

where $m(r)$ is the mass within radius r and $P' = dP/dr$.

Write down an integral expression for the total gravitational potential energy, E_{gr}. Use this to derive the "virial theorem"

$$E_{gr} = -3\langle P\rangle V ,$$

when $\langle P\rangle$ is the average pressure.

(ii) Given that the total kinetic energy, E_{kin}, of a spherically symmetric star is related to its average pressure by the formula

$$E_{kin} = \alpha\langle P\rangle V \qquad (*)$$

for constant α, use the virial theorem (stated in part (i)) to determine the condition on α needed for gravitational binding. State the relation between pressure P and "internal energy" U for an ideal gas of non-relativistic particles. What is the corresponding relation for ultra-relativistic particles? Hence show that the formula $(*)$ applies in these cases, and determine the values of α.

Why does your result imply a maximum mass for any star, whatever the source of its pressure? What is the maximum mass, approximately, for stars supported by

(a) thermal pressure,

(b) electron degeneracy pressure (White Dwarf),

(c) neutron degeneracy pressure (Neutron Star).

A White Dwarf can accrete matter from a companion star until its mass exceeds the Chandrasekar limit. Explain briefly the process by which it then evolves into a neutron star.

15F Symmetries and Groups in Physics

(i) The pions form an isospin triplet with $\pi^+ = |1,1\rangle$, $\pi^0 = |1,0\rangle$ and $\pi^- = |1,-1\rangle$, whilst the nucleons form an isospin doublet with $p = |\frac{1}{2},\frac{1}{2}\rangle$ and $n = |\frac{1}{2},-\frac{1}{2}\rangle$. Consider the isospin representation of two-particle states spanned by the basis

$$T = \{|\pi^+p\rangle,\ |\pi^+n\rangle,\ |\pi^0p\rangle,\ |\pi^0n\rangle,\ |\pi^-p\rangle,\ |\pi^-n\rangle\}\,.$$

State which irreducible representations are contained in this representation and explain why $|\pi^+p\rangle$ is an isospin eigenstate.

Using

$$I_-\,|j,m\rangle = \sqrt{(j-m+1)(j+m)}\,|j,m-1\rangle,$$

where I_- is the isospin ladder operator, write the isospin eigenstates in terms of the basis, T.

(ii) The Lie algebra $su(2)$ of generators of $SU(2)$ is spanned by the operators $\{J_+, J_-, J_3\}$ satisfying the commutator algebra $[J_+, J_-] = 2J_3$ and $[J_3, J_\pm] = \pm J_\pm$. Let Ψ_j be an eigenvector of J_3: $J_3\left(\Psi_j\right) = j\Psi_j$ such that $J_+\Psi_j = 0$. The vector space $V_j = \mathrm{span}\{J_-^n\Psi_j : n \in \mathbb{N}_0\}$ together with the action of an arbitrary $su(2)$ operator A on V_j defined by

$$J_-\left(J_-^n\Psi_j\right) = J_-^{n+1}\Psi_j\ , \qquad A\left(J_-^n\Psi_j\right) = [A, J_-]\left(J_-^{n-1}\Psi_j\right) + J_-\left(A\left(J_-^{n-1}\Psi_j\right)\right)\,,$$

forms a representation (not necessarily reducible) of $su(2)$. Show that if $J_-^n\Psi_j$ is non-trivial then it is an eigenvector of J_3 and find its eigenvalue. Given that $[J_+, J_-^n] = \alpha_n J_-^{n-1}J_3 + \beta_n J_-^{n-1}$ show that α_n and β_n satisfy

$$\alpha_n = \alpha_{n-1} + 2, \qquad \beta_n = \beta_{n-1} - \alpha_{n-1}.$$

By solving these equations evaluate $[J_+, J_-^n]$. Show that $J_+J_-^{2j+1}\Psi_j = 0$. Hence show that $J_-^{2j+1}\Psi_j$ is contained in a proper sub-representation of V_j.

16H Transport Processes

(i) Incompressible fluid of kinematic viscosity ν occupies a parallel-sided channel $0 \leqslant y \leqslant h_0$, $-\infty < x < \infty$. The wall $y = 0$ is moving parallel to itself, in the x-direction, with velocity $\mathrm{Re}\left\{Ue^{i\omega t}\right\}$, where t is time and U, ω are real constants. The fluid velocity $u(y,t)$ satisfies the equation

$$u_t = \nu u_{yy};$$

write down the boundary conditions satisfied by u.

Assuming that

$$u = \mathrm{Re}\left\{a \sinh[b(1-\eta)]e^{i\omega t}\right\},$$

where $\eta = y/h_0$, find the complex constants a, b. Calculate the velocity (in real, not complex, form) in the limit $h_0(\omega/\nu)^{1/2} \to 0$.

(ii) Incompressible fluid of viscosity μ fills the narrow gap between the rigid plane $y = 0$, which moves parallel to itself in the x-direction with constant speed U, and the rigid wavy wall $y = h(x)$, which is at rest. The length-scale, L, over which h varies is much larger than a typical value, h_0, of h.

Assume that inertia is negligible, and therefore that the governing equations for the velocity field (u, v) and the pressure p are

$$u_x + v_y = 0,\ p_x = \mu\left(u_{xx} + u_{yy}\right), p_y = \mu\left(v_{xx} + v_{yy}\right).$$

Use scaling arguments to show that these equations reduce approximately to

$$p_x = \mu u_{yy}, \quad p_y = 0.$$

Hence calculate the velocity $u(x,y)$, the flow rate

$$Q = \int_0^h u dy,$$

and the viscous shear stress exerted by the fluid on the plane wall,

$$\tau = -\mu u_y|_{y=0}$$

in terms of p_x, h, U and μ.

Now assume that $h = h_0(1 + \epsilon \sin kx)$, where $\epsilon \ll 1$ and $kh_0 \ll 1$, and that p is periodic in x with wavelength $2\pi/k$. Show that

$$Q = \frac{h_0 U}{2}\left(1 - \frac{3}{2}\epsilon^2 + O\left(\epsilon^4\right)\right)$$

and calculate τ correct to $O\left(\epsilon^2\right)$. Does increasing the amplitude ϵ of the corrugation cause an increase or a decrease in the force required to move the plane $y = 0$ at the chosen speed U?

17H Mathematical Methods

(i) The function $y(x)$ satisfies the differential equation

$$y'' + by' + cy = 0 \ , \quad 0 < x < 1 \ ,$$

where b and c are constants, with boundary conditions $y(0) = 0, \ y'(0) = 1$. By integrating this equation or otherwise, show that y must also satisfy the integral equation

$$y(x) = g(x) + \int_0^1 K(x,t) y(t) dt \ ,$$

and find the functions $g(x)$ and $K(x,t)$.

(ii) Solve the integral equation

$$\varphi(x) = 1 + \lambda^2 \int_0^x (x-t)\varphi(t) dt \ , \quad x > 0 \ , \quad \lambda \text{ real} \ ,$$

by finding an ordinary differential equation satisfied by $\varphi(x)$ together with boundary conditions.

Now solve the integral equation by the method of successive approximations and show that the two solutions are the same.

18K Nonlinear Waves

(i) The so-called breather solution of the sine-Gordon equation is

$$\phi(x,t) = 4\tan^{-1}\left(\frac{(1-\lambda^2)^{\frac{1}{2}}}{\lambda} \frac{\sin \lambda t}{\cosh(1-\lambda^2)^{\frac{1}{2}} x}\right), \quad 0 < \lambda < 1.$$

Describe qualitatively the behaviour of $\phi(x,t)$, for $\lambda \ll 1$, when $|x| \gg \ln(2/\lambda)$, when $|x| \ll 1$, and when $\cosh x \approx \frac{1}{\lambda} |\sin \lambda t|$. Explain how this solution can be interpreted in terms of motion of a kink and an antikink. Estimate the greatest separation of the kink and antikink.

(ii) The field $\psi(x,t)$ obeys the nonlinear wave equation

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} + \frac{dU}{d\psi} = 0,$$

where the potential U has the form

$$U(\psi) = \frac{1}{2}(\psi - \psi^3)^2.$$

Show that $\psi = 0$ and $\psi = 1$ are stable constant solutions.

Find a steady wave solution $\psi = f(x - vt)$ satisfying the boundary conditions $\psi \to 0$ as $x \to -\infty$, $\psi \to 1$ as $x \to \infty$. What constraint is there on the velocity v?

19K Numerical Analysis

(i) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

is discretized by the finite-difference method

$$u_m^{n+1} - \frac{1}{2}(\mu - \alpha)(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \frac{1}{2}(\mu + \alpha)(u_{m-1}^n - 2u_m^n + u_{m+1}^n),$$

where $u_m^n \approx u(m\Delta x, n\Delta t), \mu = \Delta t/(\Delta x)^2$ and α is a constant. Derive the order of magnitude (as a power of Δx) of the local error for different choices of α.

(ii) Investigate the stability of the above finite-difference method for different values of α by the Fourier technique.