UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II Alternative A

Monday 4 June 2001 1.30 to 4.30

PAPER 1

Before you begin read these instructions carefully.

Each question is divided into Part ***(i)*** *and Part* ***(ii)****, which may or may not be related. Candidates may attempt either or both Parts of any question,* **but** *must not attempt Parts from more than* ***SIX*** *questions.*

The number of marks for each question is the same, with Part ***(ii)*** *of each question carrying twice as many marks as Part* ***(i)****. Additional credit will be given for a substantially complete answer to either part.*

Begin each answer on a separate sheet.

Write legibly and on only **one** *side of the paper.*

At the end of the examination:

Tie your answers in separate bundles, marked ***A, B, C, . . . , L*** *according to the letter affixed to each question. (For example,* ***4C, 9C*** *should be in one bundle and* ***12E, 13E*** *in another bundle.)*

Attach a completed cover sheet to each bundle.

Complete a master cover sheet listing ***all*** *Parts of* ***all*** *questions attempted.*

It is essential that every cover sheet bear the candidate's examination number and desk number.

1D Markov Chains

(i) Let $X = (X_n : 0 \leqslant n \leqslant N)$ be an irreducible Markov chain on the finite state space S with transition matrix $P = (p_{ij})$ and invariant distribution π. What does it mean to say that X is reversible in equilibrium?

Show that X is reversible in equilibrium if and only if $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$.

(ii) A finite connected graph G has vertex set V and edge set E, and has neither loops nor multiple edges. A particle performs a random walk on V, moving at each step to a randomly chosen neighbour of the current position, each such neighbour being picked with equal probability, independently of all previous moves. Show that the unique invariant distribution is given by $\pi_v = d_v/(2|E|)$ where d_v is the degree of vertex v.

A rook performs a random walk on a chessboard; at each step, it is equally likely to make any of the moves which are legal for a rook. What is the mean recurrence time of a corner square. (You should give a clear statement of any general theorem used.)

[*A chessboard is an* 8×8 *square grid. A legal move is one of any length parallel to the axes.*]

2H Principles of Dynamics

(i) Show that Newton's equations in Cartesian coordinates, for a system of N particles at positions $\mathbf{x}_i(t), i = 1, 2 \ldots N$, in a potential $V(\mathbf{x}, t)$, imply Lagrange's equations in a generalised coordinate system

$$q_j = q_j(\mathbf{x}_i, t) \quad , \quad j = 1, 2 \ldots 3N;$$

that is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \quad , \quad j = 1, 2 \ldots 3N,$$

where $L = T - V$, $T(q, \dot{q}, t)$ being the total kinetic energy and $V(q, t)$ the total potential energy.

(ii) Consider a light rod of length L, free to rotate in a vertical plane (the xz plane), but with one end P forced to move in the x-direction. The other end of the rod is attached to a heavy mass M upon which gravity acts in the negative z direction.

(a) Write down the Lagrangian for the system.

(b) Show that, if P is stationary, the rod has two equilibrium positions, one stable and the other unstable.

(c) The end at P is now forced to move with constant acceleration, $\ddot{x} = A$. Show that, once more, there is one stable equilibrium value of the angle the rod makes with the vertical, and find it.

3A Functional Analysis

(i) Define the adjoint of a bounded, linear map $u : H \to H$ on the Hilbert space H. Find the adjoint of the map

$$u : H \to H \; ; \quad x \mapsto \phi(x)a$$

where $a, b \in H$ and $\phi \in H^*$ is the linear map $x \mapsto \langle b, x \rangle$.

Now let J be an **incomplete** inner product space and $u : J \to J$ a bounded, linear map. Is it always true that there is an adjoint $u^* : J \to J$?

(ii) Let $\mathcal{H}$ be the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ on the unit disc $\mathbb{D}$ for which

$$\int\int_{\mathbb{D}} |f(z)|^2 \, dx \, dy \; < \; \infty \qquad (z = x + iy).$$

You may assume that this is a Hilbert space for the inner product:

$$\langle f, g \rangle = \int\int_{\mathbb{D}} \overline{f(z)} g(z) \, dx \, dy \; .$$

Show that the functions $u_k : z \mapsto \alpha_k z^k$ $(k = 0, 1, 2, \ldots)$ form an orthonormal sequence in $\mathcal{H}$ when the constants α_k are chosen appropriately.

Prove carefully that every function $f \in \mathcal{H}$ can be written as the sum of a convergent series $\sum_{k=0}^{\infty} f_k u_k$ in $\mathcal{H}$ with $f_k \in \mathbb{C}$.

For each smooth curve γ in the disc $\mathbb{D}$ starting from 0, prove that

$$\phi : \mathcal{H} \to \mathbb{C} \; ; \quad f \mapsto \int_\gamma f(z) \, dz$$

is a continuous, linear map. Show that the norm of ϕ satisfies

$$||\phi||^2 = \frac{1}{\pi} \log \left(\frac{1}{1 - |w|^2} \right) \; ,$$

where w is the endpoint of γ.

4C Groups, Rings and Fields

(i) Define the notion of a Sylow p-subgroup of a finite group G, and state a theorem concerning the number of them and the relation between them.

(ii) Show that any group of order 30 has a non-trivial normal subgroup. Is it true that every group of order 30 is commutative?

5J Electromagnetism

(i) Write down the two Maxwell equations that govern steady magnetic fields. Show that the boundary conditions satisfied by the magnetic field on either side of a current sheet, $\mathbf{J}$, with unit normal to the sheet $\mathbf{n}$, are

$$\mathbf{n} \wedge \mathbf{B}_2 - \mathbf{n} \wedge \mathbf{B}_1 = \mu_0 \mathbf{J}.$$

State without proof the force per unit area on $\mathbf{J}$.

(ii) Conducting gas occupies the infinite slab $0 \leqslant x \leqslant a$. It carries a steady current $\mathbf{j} = (0, 0, j)$ and a magnetic field $\mathbf{B} = (0, B, 0)$ where $\mathbf{j}, \mathbf{B}$ depend only on x. The pressure is $p(x)$. The equation of hydrostatic equilibrium is $\nabla p = \mathbf{j} \wedge \mathbf{B}$. Write down the equations to be solved in this case. Show that $p + (1/2\mu_0)B^2$ is independent of x. Using the suffixes 1,2 to denote values at $x = 0, a$, respectively, verify that your results are in agreement with those of Part (i) in the case of $a \to 0$.

Suppose that

$$j(x) = \frac{\pi j_0}{2a} \sin\left(\frac{\pi x}{a}\right), \quad B_1 = 0, \quad p_2 = 0.$$

Find $B(x)$ everywhere in the slab.

6K Dynamics of Differential Equations

(i) Given a differential equation $\dot{x} = f(x)$ for $x \in \mathbb{R}^n$, explain what it means to say that the solution is given by a flow $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Define the orbit, $o(x)$, through a point x and the ω-limit set, $\omega(x)$, of x. Define also a homoclinic orbit to a fixed point x_0. Sketch a flow in $\mathbb{R}^2$ with a homoclinic orbit, and identify (without detailed justification) the ω-limit sets $\omega(x)$ for each point x in your diagram.

(ii) Consider the differential equations

$$\dot{x} = zy, \qquad \dot{y} = -zx, \qquad \dot{z} = -z^2\,.$$

Transform the equations to polar coordinates (r, θ) in the (x, y) plane. Solve the equation for z to find $z(t)$, and hence find $\theta(t)$. Hence, or otherwise, determine (with justification) the ω-limit set for all points $(x_0, y_0, z_0) \in \mathbb{R}^3$.

7B Logic, Computation and Set Theory

(i) What is the *Halting Problem*? What is an *unsolvable* problem?

(ii) Prove that the Halting Problem is unsolvable. Is it decidable whether or not a machine halts with input zero?

8A Graph Theory

(i) Show that any graph G with minimal degree δ contains a cycle of length at least $\delta + 1$. Give examples to show that, for each possible value of δ, there is a graph with minimal degree δ but no cycle of length greater than $\delta + 1$.

(ii) Let K_N be the complete graph with N vertices labelled $v_1, v_2, \ldots, v_N$. Prove, from first principles, that there are N^{N-2} different spanning trees in K_N. In how many of these spanning trees does the vertex v_1 have degree 1?

A spanning tree in K_N is chosen at random, with each of the N^{N-2} trees being equally likely. Show that the average number of vertices of degree 1 in the random tree is approximately N/e when N is large.

Find the average degree of vertices in the random tree.

9C Number Theory

(i) Describe Euclid's algorithm.

Find, in the RSA algorithm, the deciphering key corresponding to the enciphering key $7, 527$.

(ii) Explain what is meant by a primitive root modulo an odd prime p.

Show that, if g is a primitive root modulo p, then all primitive roots modulo p are given by g^m, where $1 \leqslant m < p$ and $(m, p-1) = 1$.

Verify, by Euler's criterion, that 3 is a primitive root modulo 17. Hence find all primitive roots modulo 17.

10A Coding and Cryptography

(i) Explain briefly how and why a signature scheme is used. Describe the el Gamal scheme.

(ii) Define a cyclic code. Define the generator of a cyclic code and show that it exists. Prove a necessary and sufficient condition for a polynomial to be the generator of a cyclic code of length n.

What is the BCH code? Show that the BCH code associated with $\{\beta, \beta^2\}$, where β is a root of $X^3 + X + 1$ in an appropriate field, is Hamming's original code.

11D Stochastic Financial Models

(i) The price of the stock in the binomial model at time r, $1 \leqslant r \leqslant n$, is $S_r = S_0 \prod_{j=1}^{r} Y_j$, where $Y_1, Y_2, \ldots, Y_n$ are independent, identically-distributed random variables with $\mathbb{P}(Y_1 = u) = p = 1 - \mathbb{P}(Y_1 = d)$ and the initial price S_0 is a constant. Denote the fixed interest rate on the bank account by ρ, where $u > 1 + \rho > d > 0$, and let the discount factor $\alpha = 1/(1+\rho)$. Determine the unique value $p = \overline{p}$ for which the sequence $\{\alpha^r S_r, \, 0 \leqslant r \leqslant n\}$ is a martingale.

Explain briefly the significance of $\overline{p}$ for the pricing of contingent claims in the model.

(ii) Let T_a denote the first time that a standard Brownian motion reaches the level $a > 0$. Prove that for $t > 0$,

$$\mathbb{P}(T_a \leqslant t) = 2\left[1 - \Phi\left(a/\sqrt{t}\right)\right],$$

where Φ is the standard normal distribution function.

Suppose that A_t and B_t represent the prices at time t of two different stocks with initial prices 1 and 2, respectively; the prices evolve so that they may be represented as $A_t = e^{\sigma_1 X_t + \mu t}$ and $B_t = 2e^{\sigma_2 Y_t + \mu t}$, respectively, where $\{X_t\}_{t \geqslant 0}$ and $\{Y_t\}_{t \geqslant 0}$ are independent standard Brownian motions and σ_1, σ_2 and μ are constants. Let T denote the first time, if ever, that the prices of the two stocks are the same. Determine $\mathbb{P}(T \leqslant t)$, for $t > 0$.

12E Principles of Statistics

(i) What are the main approaches by which prior distributions are specified in Bayesian inference?

Define the risk function of a decision rule d. Given a prior distribution, define what is meant by a Bayes decision rule and explain how this is obtained from the posterior distribution.

(ii) Dashing late into King's Cross, I discover that Harry must have already boarded the Hogwarts Express. I must therefore make my own way onto platform nine and three-quarters. Unusually, there are two guards on duty, and I will ask one of them for directions. It is safe to assume that one guard is a Wizard, who will certainly be able to direct me, and the other a Muggle, who will certainly not. But which is which? Before choosing one of them to ask for directions to platform nine and three-quarters, I have just enough time to ask one of them "Are you a Wizard?", and on the basis of their answer I must make my choice of which guard to ask for directions. I know that a Wizard will answer this question truthfully, but that a Muggle will, with probability $\frac{1}{3}$, answer it untruthfully.

Failure to catch the Hogwarts Express results in a loss which I measure as 1000 galleons, there being no loss associated with catching up with Harry on the train.

Write down an exhaustive set of non-randomised decision rules for my problem and, by drawing the associated risk set, determine my minimax decision rule.

My prior probability is $\frac{2}{3}$ that the guard I ask "Are you a Wizard?" is indeed a Wizard. What is my Bayes decision rule?

13E Computational Statistics and Statistical Modelling

(i) Assume that the n-dimensional observation vector Y may be written as

$$Y = X\beta + \epsilon \quad ,$$

where X is a given $n \times p$ matrix of rank p, β is an unknown vector, and

$$\epsilon \sim N_n(0, \sigma^2 I).$$

Let $Q(\beta) = (Y - X\beta)^T (Y - X\beta)$. Find $\widehat{\beta}$, the least-squares estimator of β, and show that

$$Q(\widehat{\beta}) = Y^T (I - H) Y \quad ,$$

where H is a matrix that you should define.

(ii) Show that $\sum_i H_{ii} = p$. Show further for the special case of

$$Y_i = \beta_1 + \beta_2 x_i + \beta_3 z_i + \epsilon_i, \quad 1 \leqslant i \leqslant n,$$

where $\Sigma x_i = 0, \quad \Sigma z_i = 0$, that

$$H \quad = \quad \frac{1}{n}\mathbf{1}\mathbf{1}^T + a x x^T + b(x z^T + z x^T) + c z z^T \quad ;$$

here, $\mathbf{1}$ is a vector of which every element is one, and a, b, c, are constants that you should derive.

Hence show that, if $\widehat{Y} = X\widehat{\beta}$ is the vector of fitted values, then

$$\frac{1}{\sigma^2}\mathrm{var}(\widehat{Y}_i) = \frac{1}{n} + a x_i^2 + 2b x_i z_i + c z_i^2, \quad 1 \leqslant i \leqslant n.$$

14F Quantum Physics

(i) A spinless quantum mechanical particle of mass m moving in two dimensions is confined to a square box with sides of length L. Write down the energy eigenfunctions for the particle and the associated energies.

Show that, for large L, the number of states in the energy range $E \to E + dE$ is $\rho(E)dE$, where

$$\rho(E) = \frac{mL^2}{2\pi\hbar^2}.$$

(ii) If, instead, the particle is an electron with magnetic moment μ moving in an external magnetic field, H, show that

$$\begin{aligned}\rho(E) =& \frac{mL^2}{2\pi\hbar^2}, \qquad & -\mu H < E < \mu H \\ =& \frac{mL^2}{\pi\hbar^2}, \qquad & \mu H < E.\end{aligned}$$

Let there be N electrons in the box. Explain briefly how to construct the ground state of the system. Let E_F be the Fermi energy. Show that when $E_F > \mu H$,

$$N = \frac{mL^2}{\pi\hbar^2} E_F.$$

Show also that the magnetic moment, M, of the system in the ground state is

$$M = \frac{\mu^2 mL^2}{\pi\hbar^2} H,$$

and that the ground state energy is

$$\frac{1}{2}\frac{\pi\hbar^2}{mL^2} N^2 - \frac{1}{2} MH.$$

15J General Relativity

(i) The metric of any two-dimensional curved space, rotationally symmetric about a point P, can by suitable choice of coordinates be written locally in the form

$$ds^2 = e^{2\phi(r)}(dr^2 + r^2 d\theta^2),$$

where $r = 0$ at P, $r > 0$ away from P, and $0 \leqslant \theta < 2\pi$. Labelling the coordinates as $(x^1, x^2) = (r, \theta)$, show that the Christoffel symbols $\Gamma^1_{12}, \Gamma^2_{11}$ and Γ^2_{22} are each zero, and compute the non-zero Christoffel symbols $\Gamma^1_{11}, \Gamma^1_{22}$ and $\Gamma^2_{12} = \Gamma^2_{21}$.

The Ricci tensor R_{ab} $(a, b = 1, 2)$ is defined by

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{cd}\Gamma^d_{ab} - \Gamma^d_{ac}\Gamma^c_{bd},$$

where a comma denotes a partial derivative. Show that $R_{12} = 0$ and that

$$R_{11} = -\phi'' - r^{-1}\phi', \qquad R_{22} = r^2 R_{11}.$$

(ii) Suppose further that, in a neighbourhood of P, the Ricci scalar R takes the constant value -2. Find a second order differential equation, which you should denote by $(*)$, for $\phi(r)$.

This space of constant Ricci scalar can, by a suitable coordinate transformation $r \to \chi(r)$, leaving θ invariant, be written locally as

$$ds^2 = d\chi^2 + \sinh^2 \chi d\theta^2$$

By studying this coordinate transformation, or otherwise, find $\cosh\chi$ and $\sinh\chi$ in terms of r (up to a constant of integration). Deduce that

$$e^{\phi(r)} = \frac{2A}{(1 - A^2 r^2)} \quad , \quad (0 \leqslant Ar < 1),$$

where A is a positive constant and verify that your equation $(*)$ for ϕ holds.

[*Note that*

$$\int \frac{d\chi}{\sinh\chi} = \text{const.} + \frac{1}{2}\log\bigl(\cosh\chi - 1\bigr) - \frac{1}{2}\log\bigl(\cosh\chi + 1\bigr).\Bigr]$$

16J Statistical Physics and Cosmology

(i) Introducing the concept of a co-moving distance co-ordinate, explain briefly how the velocity of a galaxy in an isotropic and homogeneous universe is determined by the scale factor $a(t)$. How is the scale factor related to the Hubble constant H_0?

A homogeneous and isotropic universe has an energy density $\rho(t)c^2$ and a pressure $P(t)$. Use the relation $dE = -PdV$ to derive the "fluid equation"

$$\dot{\rho} = -3\left(\rho + \frac{P}{c^2}\right)\left(\frac{\dot{a}}{a}\right),$$

where the overdot indicates differentiation with respect to time, t. Given that $a(t)$ satisfies the "acceleration equation"

$$\ddot{a} = -\frac{4\pi G}{3}\, a\left(\rho + \frac{3P}{c^2}\right),$$

show that the quantity

$$k = c^{-2}\left(\frac{8\pi G}{3}\rho a^2 \;- \dot{a}^2\right)$$

is time-independent.

The pressure P is related to ρ by the "equation of state"

$$P = \sigma\rho c^2, \quad |\sigma| < 1\ .$$

Given that $a(t_0) = 1$, find $a(t)$ for $k = 0$, and hence show that $a(0) = 0$.

(ii) What is meant by the expression "the Hubble time"?

Assuming that $a(0) = 0$ and $a(t_0) = 1$, where t_0 is the time now (of the current cosmological era), obtain a formula for the radius R_0 of the observable universe.

Given that

$$a(t) \quad = \quad \left(\frac{t}{t_0}\right)^{\alpha}$$

for constant α, find the values of α for which R_0 is finite. Given that R_0 is finite, show that the age of the universe is less than the Hubble time. Explain briefly, and qualitatively, why this result is to be expected as long as

$$\rho + 3\frac{P}{c^2} > 0.$$

17F Symmetries and Groups in Physics

(i) Let $h : G \to G'$ be a surjective homomorphism between two groups, G and G'. If $D' : G' \to GL(\mathbb{C}^n)$ is a representation of G', show that $D(g) = D'(h(g))$ for $g \in G$ is a representation of G and, if D' is irreducible, show that D is also irreducible. Show further that $\widetilde{D}(\widetilde{g}) = D'\left(\widetilde{h}(\widetilde{g})\right)$ is a representation of $G/\ker(h)$, where $\widetilde{h}(\widetilde{g}) = h(g)$ for $g \in G$ and $\widetilde{g} \in G/\ker(h)$ (with $g \in \widetilde{g}$). Deduce that the characters $\chi, \widetilde{\chi}, \chi'$ of $D, \widetilde{D}, D'$, respectively, satisfy

$$\chi(g) = \widetilde{\chi}(\widetilde{g}) = \chi'(h(g)) .$$

(ii) D_4 is the symmetry group of rotations and reflections of a square. If c is a rotation by $\pi/2$ about the centre of the square and b is a reflection in one of its symmetry axes, then $D_4 = \{e, c, c^2, c^3, b, bc, bc^2, bc^3\}$. Given that the conjugacy classes are $\{e\}$ $\{c^2\}$, $\{c, c^3\}$ $\{b, bc^2\}$ and $\{bc, bc^3\}$ derive the character table of D_4.

Let H_0 be the Hamiltonian of a particle moving in a central potential. The $SO(3)$ symmetry ensures that the energy eigenvalues of H_0 are the same for all the angular momentum eigenstates in a given irreducible $SO(3)$ representation. Therefore, the energy eigenvalues of H_0 are labelled E_{nl} with $n \in \mathbb{N}$ and $l \in \mathbb{N}_0$, $l < n$. Assume now that in a crystal the symmetry is reduced to a D_4 symmetry by an additional term H_1 of the total Hamiltonian, $H = H_0 + H_1$. Find how the H_0 eigenstates in the $SO(3)$ irreducible representation with $l = 2$ (the D-wave orbital) decompose into irreducible representations of H. You may assume that the character, $g(\theta)$, of a group element of $SO(3)$, in a representation labelled by l is given by

$$\chi(g_\theta) = 1 + 2\cos\theta + 2\cos(2\theta) + \ldots + 2\cos(l\theta),$$

where θ is a rotation angle and $l(l+1)$ is the eigenvalue of the total angular momentum, $\mathbf{L}^2$.

18H Transport Processes

(i) The diffusion equation for a spherically-symmetric concentration field $C(r,t)$ is

$$C_t = \frac{D}{r^2}\left(r^2 C_r\right)_r\,, \tag{1}$$

where r is the radial coordinate. Find and sketch the similarity solution to (1) which satisfies $C \to 0$ as $r \to \infty$ and $\int_0^\infty 4\pi r^2 C(r,t)dr = M =$ constant, assuming it to be of the form

$$C = \frac{M}{(Dt)^a}F(\eta), \quad \eta = \frac{r}{(Dt)^b},$$

where a and b are numbers to be found.

(ii) A two-dimensional piece of heat-conducting material occupies the region $a \leqslant r \leqslant b,\ -\pi/2 \leqslant \theta \leqslant \pi/2$ (in plane polar coordinates). The surfaces $r = a,\ \theta = -\pi/2,\ \theta = \pi/2$ are maintained at a constant temperature T_1; at the surface $r = b$ the boundary condition on temperature $T(r,\theta)$ is

$$T_r + \beta T = 0,$$

where $\beta > 0$ is a constant. Show that the temperature, which satisfies the steady heat conduction equation

$$T_{rr} + \frac{1}{r}T_r + \frac{1}{r^2}T_{\theta\theta} = 0,$$

is given by a Fourier series of the form

$$\frac{T}{T_1} = K + \sum_{n=0}^{\infty}\cos\left(\alpha_n\theta\right)\left[A_n\left(\frac{r}{a}\right)^{2n+1} + B_n\left(\frac{a}{r}\right)^{2n+1}\right],$$

where $K,\ \alpha_n,\ A_n,\ B_n$ are to be found.

In the limits $a/b \ll 1$ and $\beta b \ll 1$, show that

$$\int_{-\pi/2}^{\pi/2} T_r r d\theta \approx -\pi\beta b T_1,$$

given that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Explain how, in these limits, you could have obtained this result much more simply.

19L Theoretical Geophysics

(i) From the surface of a flat Earth, an explosive source emits P-waves downward into a horizontal homogeneous elastic layer of uniform thickness h and P-wave speed α_1 overlying a lower layer of infinite depth and P-wave speed α_2, where $\alpha_2 > \alpha_1$. A line of seismometers on the surface records the travel time t as a function of distance x from the source for the various arrivals along different ray paths.

Sketch the ray paths associated with the direct, reflected and head waves arriving at a given position. Calculate the travel times $t(x)$ of the direct and reflected waves, and sketch the corresponding travel-time curves. Hence explain how to estimate α_1 and h from the recorded arrival times. Explain briefly why head waves are only observed beyond a minimum distance x_c from the source and why they have a travel-time curve of the form $t = t_c + (x - x_c)/\alpha_2$ for $x > x_c$.
[*You need not calculate x_c or t_c.*]

(ii) A plane SH-wave in a homogeneous elastic solid has displacement proportional to $\exp[i(kx+mz-\omega t)]$. Express the slowness vector $\mathbf{s}$ in terms of the wavevector $\mathbf{k} = (k, 0, m)$ and ω. Deduce an equation for m in terms of k, ω and the S-wave speed β.

A homogeneous elastic layer of uniform thickness h, S-wave speed β_1 and shear modulus μ_1 has a stress-free surface $z = 0$ and overlies a lower layer of infinite depth, S-wave speed β_2 ($> \beta_1$) and shear modulus μ_2. Find the vertical structure of Love waves with displacement proportional to $\exp[i(kx - \omega t)]$, and show that the horizontal phase speed c obeys

$$\tan\left[\left(\frac{1}{\beta_1^2} - \frac{1}{c^2}\right)^{1/2} \omega h\right] = \frac{\mu_2}{\mu_1}\left(\frac{1/c^2 - 1/\beta_2^2}{1/\beta_1^2 - 1/c^2}\right)^{1/2} .$$

By sketching both sides of the equation as a function of c in $\beta_1 \leqslant c \leqslant \beta_2$ show that at least one mode exists for every value of ω.

20K Numerical Analysis

(i) Let A be a symmetric $n \times n$ matrix such that

$$A_{k,k} > \sum_{\substack{l=1 \\ l \neq k}}^{n} |A_{k,l}| \qquad 1 \leqslant k \leqslant n.$$

Prove that it is positive definite.

(ii) Prove that both Jacobi and Gauss-Seidel methods for the solution of the linear system $A\mathbf{x} = \mathbf{b}$, where the matrix A obeys the conditions of (i), converge.

[*You may quote the Householder-John theorem without proof.*]