UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 8 June, 2012 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, ..., H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Let V be a complex vector space with basis $\{e_1, \ldots, e_n\}$. Define $T : V \to V$ by $T(e_i) = e_i - e_{i+1}$ for $i < n$ and $T(e_n) = e_n - e_1$. Show that T is diagonalizable and find its eigenvalues. [You may use any theorems you wish, as long as you state them clearly.]

2G Groups, Rings and Modules

An *idempotent* element of a ring R is an element e satisfying $e^2 = e$. A *nilpotent* element is an element e satisfying $e^N = 0$ for some $N \geqslant 0$.

Let $r \in R$ be non-zero. In the ring $R[X]$, can the polynomial $1 + rX$ be (i) an idempotent, (ii) a nilpotent? Can $1 + rX$ satisfy the equation $(1 + rX)^3 = (1 + rX)$? Justify your answers.

3E Analysis II

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be a bilinear function. Show that f is differentiable at any point in $\mathbb{R}^n \times \mathbb{R}^m$ and find its derivative.

4E Complex Analysis

Let $h : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with $h(i) \neq h(-i)$. Does there exist a holomorphic function f defined in $|z| < 1$ for which $f'(z) = \dfrac{h(z)}{1+z^2}$? Does there exist a holomorphic function f defined in $|z| > 1$ for which $f'(z) = \dfrac{h(z)}{1+z^2}$? Justify your answers.

5D Methods

Show that the general solution of the wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$$

can be written in the form

$$y(x,t) = f(x - ct) + g(x + ct)\,.$$

Hence derive the solution $y(x,t)$ subject to the initial conditions

$$y(x,0) = 0, \qquad \frac{\partial y}{\partial t}(x,0) = \psi(x)\,.$$

6C Quantum Mechanics

In terms of quantum states, what is meant by *energy degeneracy*?

A particle of mass m is confined within the box $0 < x < a$, $0 < y < a$ and $0 < z < c$. The potential vanishes inside the box and is infinite outside. Find the allowed energies by considering a stationary state wavefunction of the form

$$\chi(x, y, z) = X(x)\, Y(y)\, Z(z)\,.$$

Write down the normalised ground state wavefunction. Assuming that $c < a < \sqrt{2}c$, give the energies of the first three excited states.

7B Electromagnetism

Define the notions of magnetic flux, electromotive force and resistance, in the context of a single closed loop of wire. Use the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

to derive Faraday's law of induction for the loop, assuming the loop is at rest.

Suppose now that the magnetic field is $\mathbf{B} = (0, 0, B \tanh t)$ where B is a constant, and that the loop of wire, with resistance R, is a circle of radius a lying in the (x, y) plane. Calculate the current in the wire as a function of time.

Explain briefly why, even in a time-independent magnetic field, an electromotive force may be produced in a loop of wire that moves through the field, and state the law of induction in this situation.

8D Numerical Analysis

State the Dahlquist equivalence theorem regarding convergence of a multistep method.

The multistep method, with a real parameter a,

$$y_{n+3} + (2a - 3)(y_{n+2} - y_{n+1}) - y_n = ha\,(f_{n+2} - f_{n+1})$$

is of order 2 for any a, and also of order 3 for $a = 6$. Determine all values of a for which the method is convergent, and find the order of convergence.

9H Markov Chains

Let $(X_n)_{n \geqslant 0}$ be an irreducible Markov chain with $p_{ij}^{(n)} = P(X_n = j \mid X_0 = i)$. Define the meaning of the statements:

(i) state i is transient,

(ii) state i is aperiodic.

Give a criterion for transience that can be expressed in terms of the probabilities $(p_{ii}^{(n)},\ n = 0, 1, \ldots)$.

Prove that if a state i is transient then all states are transient.

Prove that if a state i is aperiodic then all states are aperiodic.

Suppose that $p_{ii}^{(n)} = 0$ unless n is divisible by 3. Given any other state j, prove that $p_{jj}^{(n)} = 0$ unless n is divisible by 3.

SECTION II

10F Linear Algebra

Let V be a finite-dimensional real vector space of dimension n. A bilinear form $B : V \times V \to \mathbb{R}$ is *nondegenerate* if for all $\mathbf{v} \neq 0$ in V, there is some $\mathbf{w} \in V$ with $B(\mathbf{v}, \mathbf{w}) \neq 0$. For $\mathbf{v} \in V$, define $\langle \mathbf{v} \rangle^{\perp} = \{\mathbf{w} \in V \mid B(\mathbf{v}, \mathbf{w}) = 0\}$. Assuming B is nondegenerate, show that $V = \langle \mathbf{v} \rangle \oplus \langle \mathbf{v} \rangle^{\perp}$ whenever $B(\mathbf{v}, \mathbf{v}) \neq 0$.

Suppose that B is a nondegenerate, symmetric bilinear form on V. Prove that there is a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V with $B(\mathbf{v}_i, \mathbf{v}_j) = 0$ for $i \neq j$. [If you use the fact that symmetric matrices are diagonalizable, you must prove it.]

Define the signature of a quadratic form. Explain how to determine the signature of the quadratic form associated to B from the basis you constructed above.

A linear subspace $V' \subset V$ is said to be *isotropic* if $B(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{v}, \mathbf{w} \in V'$. Show that if B is nondegenerate, the maximal dimension of an isotropic subspace of V is $(n - |\sigma|)/2$, where σ is the signature of the quadratic form associated to B.

11G Groups, Rings and Modules

Let R be a commutative ring with unit 1. Prove that an R-module is finitely generated if and only if it is a quotient of a free module R^n, for some $n > 0$.

Let M be a finitely generated R-module. Suppose now I is an ideal of R, and ϕ is an R-homomorphism from M to M with the property that

$$\phi(M) \subset I \cdot M \ = \{m \in M \mid m = rm' \quad \text{with} \quad r \in I\,,\ m' \in M\}\,.$$

Prove that ϕ satisfies an equation

$$\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_1\phi + a_0 = 0$$

where each $a_j \in I$. [You may assume that if T is a matrix over R, then $\mathrm{adj}(T)\,T = \det T\,(\mathrm{id})$, with id the identity matrix.]

Deduce that if M satisfies $I \cdot M = M$, then there is some $a \in R$ satisfying

$$a - 1 \in I \quad \text{and} \quad aM = 0\,.$$

Give an example of a finitely generated $\mathbb{Z}$-module M and a proper ideal I of $\mathbb{Z}$ satisfying the hypothesis $I \cdot M = M$, and for your example, give an explicit such element a.

12E Analysis II

State and prove the Bolzano-Weierstrass theorem in $\mathbb{R}^n$. [You may assume the Bolzano-Weierstrass theorem in $\mathbb{R}$.]

Let $X \subset \mathbb{R}^n$ be a subset and let $f : X \to X$ be a mapping such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$, where d is the Euclidean distance in $\mathbb{R}^n$. Prove that if X is closed and bounded, then f is a bijection. Is this result still true if we drop the boundedness assumption on X? Justify your answer.

13F Metric and Topological Spaces

Suppose A_1 and A_2 are topological spaces. Define the product topology on $A_1 \times A_2$. Let $\pi_i : A_1 \times A_2 \to A_i$ be the projection. Show that a map $F : X \to A_1 \times A_2$ is continuous if and only if $\pi_1 \circ F$ and $\pi_2 \circ F$ are continuous.

Prove that if A_1 and A_2 are connected, then $A_1 \times A_2$ is connected.

Let X be the topological space whose underlying set is $\mathbb{R}$, and whose open sets are of the form (a, ∞) for $a \in \mathbb{R}$, along with the empty set and the whole space. Describe the open sets in $X \times X$. Are the maps $f, g : X \times X \to X$ defined by $f(x, y) = x + y$ and $g(x, y) = xy$ continuous? Justify your answers.

14A Complex Methods

State the convolution theorem for Fourier transforms.

The function $\phi(x, y)$ satisfies

$$\nabla^2 \phi = 0$$

on the half-plane $y \geqslant 0$, subject to the boundary conditions

$$\phi \to 0 \quad \text{as} \quad y \to \infty \quad \text{for all} \quad x,$$

$$\phi(x,0) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| > 1\,. \end{cases}$$

Using Fourier transforms, show that

$$\phi(x, y) = \frac{y}{\pi} \int_{-1}^{1} \frac{1}{y^2 + (x-t)^2}\,\mathrm{d}t\,,$$

and hence that

$$\phi(x, y) = \frac{1}{\pi} \left[\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{1+x}{y}\right) \right].$$

15G Geometry

Let $\Sigma \subset \mathbb{R}^3$ be a smooth closed surface. Define the *principal curvatures* κ_{max} and κ_{min} at a point $p \in \Sigma$. Prove that the Gauss curvature at p is the product of the two principal curvatures.

A point $p \in \Sigma$ is called a *parabolic point* if at least one of the two principal curvatures vanishes. Suppose $\Pi \subset \mathbb{R}^3$ is a plane and Σ is tangent to Π along a smooth closed curve $C = \Pi \cap \Sigma \subset \Sigma$. Show that C is composed of parabolic points.

Can both principal curvatures vanish at a point of C? Briefly justify your answer.

16B Variational Principles

Consider a functional

$$I = \int_a^b F(x, y, y')dx$$

where F is smooth in all its arguments, $y(x)$ is a C^1 function and $y' = \frac{dy}{dx}$. Consider the function $y(x) + h(x)$ where $h(x)$ is a small C^1 function which vanishes at a and b. Obtain formulae for the first and second variations of I about the function $y(x)$. Derive the Euler-Lagrange equation from the first variation, and state its variational interpretation.

Suppose now that

$$I = \int_0^1 (y'^2 - 1)^2 dx$$

where $y(0) = 0$ and $y(1) = \beta$. Find the Euler-Lagrange equation and the formula for the second variation of I. Show that the function $y(x) = \beta x$ makes I stationary, and that it is a (local) minimizer if $\beta > \frac{1}{\sqrt{3}}$.

Show that when $\beta = 0$, the function $y(x) = 0$ is not a minimizer of I.

17D Methods

Let $D \subset \mathbb{R}^2$ be a two-dimensional domain with boundary $S = \partial D$, and let

$$G_2 = G_2(\mathbf{r}, \mathbf{r}_0) = \frac{1}{2\pi} \log |\mathbf{r} - \mathbf{r}_0| \,,$$

where $\mathbf{r}_0$ is a point in the interior of D. From Green's second identity,

$$\int_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) d\ell = \int_D (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, da \,,$$

derive Green's third identity

$$u(\mathbf{r}_0) = \int_D G_2 \nabla^2 u \, da + \int_S \left(u \frac{\partial G_2}{\partial n} - G_2 \frac{\partial u}{\partial n} \right) d\ell \,.$$

[Here $\frac{\partial}{\partial n}$ denotes the normal derivative on S.]

Consider the Dirichlet problem on the unit disc $D_1 = \{\mathbf{r} \in \mathbb{R}^2 : |\mathbf{r}| \leqslant 1\}$:

$$\begin{aligned} \nabla^2 u = 0, \quad & \mathbf{r} \in D_1 \,, \\ u(\mathbf{r}) = f(\mathbf{r}), \quad & \mathbf{r} \in S_1 = \partial D_1 \,. \end{aligned}$$

Show that, with an appropriate function $G(\mathbf{r}, \mathbf{r}_0)$, the solution can be obtained by the formula

$$u(\mathbf{r}_0) = \int_{S_1} f(\mathbf{r}) \frac{\partial}{\partial n} G(\mathbf{r} \,, \mathbf{r}_0) \, d\ell \,.$$

State the boundary conditions on G and explain how G is related to G_2.

For $\mathbf{r}, \mathbf{r}_0 \in \mathbb{R}^2$, prove the identity

$$\left| \frac{\mathbf{r}}{|\mathbf{r}|} - \mathbf{r}_0 |\mathbf{r}| \right| = \left| \frac{\mathbf{r}_0}{|\mathbf{r}_0|} - \mathbf{r} |\mathbf{r}_0| \right| ,$$

and deduce that if the point $\mathbf{r}$ lies on the unit circle, then

$$|\mathbf{r} - \mathbf{r}_0| = |\mathbf{r}_0| |\mathbf{r} - \mathbf{r}_0^*| \,, \text{ where } \mathbf{r}_0^* = \frac{\mathbf{r}_0}{|\mathbf{r}_0|^2} \,.$$

Hence, using the method of images, or otherwise, find an expression for the function $G(\mathbf{r} \,, \mathbf{r}_0)$. [An expression for $\frac{\partial}{\partial n} G$ is not required.]

18A Fluid Dynamics

The equations governing the flow of a shallow layer of inviscid liquid of uniform depth H rotating with angular velocity $\frac{1}{2}f$ about the vertical z-axis are

$$\begin{aligned} \frac{\partial u}{\partial t} - fv &= -g\frac{\partial \eta}{\partial x}\,, \\ \frac{\partial v}{\partial t} + fu &= -g\frac{\partial \eta}{\partial y}\,, \\ \frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0\,, \end{aligned}$$

where u, v are the x- and y-components of velocity, respectively, and η is the elevation of the free surface. Show that these equations imply that

$$\frac{\partial q}{\partial t} = 0\,, \quad \text{where} \quad q = \omega - \frac{f\eta}{H} \quad \text{and} \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,.$$

Consider an initial state where there is flow in the y-direction given by

$$\begin{aligned} &u = \eta = 0, \quad -\infty < x < \infty \\ &v = \begin{cases} \frac{g}{2f}e^{2x}, & x < 0 \\ -\frac{g}{2f}e^{-2x}, & x > 0\,. \end{cases} \end{aligned}$$

Find the initial potential vorticity.

Show that when this initial state adjusts, there is a final steady state independent of y in which η satisfies

$$\frac{\partial^2 \eta}{\partial x^2} - \frac{\eta}{a^2} = \begin{cases} e^{2x}, & x < 0 \\ e^{-2x}, & x > 0\,, \end{cases}$$

where $a^2 = gH/f^2$.

In the case $a = 1$, find the final free surface elevation that is finite at large $|x|$ and which is continuous and has continuous slope at $x = 0$, and show that it is negative for all x.

19H Statistics

From each of 3 populations, n data points are sampled and these are believed to obey

$$y_{ij} = \alpha_i + \beta_i(x_{ij} - \bar{x}_i) + \epsilon_{ij}, \quad j \in \{1, \ldots, n\},\ i \in \{1, 2, 3\},$$

where $\bar{x}_i = (1/n)\sum_j x_{ij}$, the ϵ_{ij} are independent and identically distributed as $N(0, \sigma^2)$, and σ^2 is unknown. Let $\bar{y}_i = (1/n)\sum_j y_{ij}$.

(i) Find expressions for $\hat{\alpha}_i$ and $\hat{\beta}_i$, the least squares estimates of α_i and β_i.

(ii) What are the distributions of $\hat{\alpha}_i$ and $\hat{\beta}_i$?

(iii) Show that the residual sum of squares, R_1, is given by

$$R_1 = \sum_{i=1}^{3} \left[\sum_{j=1}^{n} (y_{ij} - \bar{y}_i)^2 - \hat{\beta}_i^2 \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)^2 \right].$$

Calculate R_1 when $n = 9$, $\{\hat{\alpha}_i\}_{i=1}^3 = \{1.6, 0.6, 0.8\}$, $\{\hat{\beta}_i\}_{i=1}^3 = \{2, 1, 1\}$,

$$\left\{ \sum_{j=1}^{9} (y_{ij} - \bar{y}_i)^2 \right\}_{i=1}^{3} = \{138, 82, 63\}, \quad \left\{ \sum_{j=1}^{9} (x_{ij} - \bar{x}_i)^2 \right\}_{i=1}^{3} = \{30, 60, 40\}.$$

(iv) H_0 is the hypothesis that $\alpha_1 = \alpha_2 = \alpha_3$. Find an expression for the maximum likelihood estimator of α_1 under the assumption that H_0 is true. Calculate its value for the above data.

(v) Explain (stating without proof any relevant theory) the rationale for a statistic which can be referred to an F distribution to test H_0 against the alternative that it is not true. What should be the degrees of freedom of this F distribution? What would be the outcome of a size 0.05 test of H_0 with the above data?

20H Optimization

Describe the Ford-Fulkerson algorithm.

State conditions under which the algorithm is guaranteed to terminate in a finite number of steps. Explain why it does so, and show that it finds a maximum flow. [You may assume that the value of a flow never exceeds the value of any cut.]

In a football league of n teams the season is partly finished. Team i has already won w_i matches. Teams i and j are to meet in m_{ij} further matches. Thus the total number of remaining matches is $M = \sum_{i<j} m_{ij}$. Assume there will be no drawn matches. We wish to determine whether it is possible for the outcomes of the remaining matches to occur in such a way that at the end of the season the numbers of wins by the teams are $(x_1, \ldots, x_n)$.

Invent a network flow problem in which the maximum flow from source to sink equals M if and only if $(x_1, \ldots, x_n)$ is a feasible vector of final wins.

Illustrate your idea by answering the question of whether or not $x = (7, 5, 6, 6)$ is a possible profile of total end-of-season wins when $n = 4$, $w = (1, 2, 3, 4)$, and $M = 14$ with

$$(m_{ij}) = \begin{pmatrix} - & 2 & 2 & 2 \\ 2 & - & 1 & 1 \\ 2 & 1 & - & 6 \\ 2 & 1 & 6 & - \end{pmatrix}.$$

END OF PAPER