UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 7 June, 2012 9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

*Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most **four** questions from Section I and at most **six** questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in separate bundles labelled **A, B, . . . , H** according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Groups, Rings and Modules

What is a *Euclidean domain*?

Giving careful statements of any general results you use, show that in the ring $\mathbb{Z}[\sqrt{-3}]$, 2 is irreducible but not prime.

2E Analysis II

Let $C[0,1]$ be the set of continuous real-valued functions on $[0,1]$ with the uniform norm. Suppose $T : C[0,1] \to C[0,1]$ is defined by

$$T(f)(x) = \int_0^x f(t^3)\, dt\,,$$

for all $x \in [0,1]$ and $f \in C[0,1]$. Is T a contraction mapping? Does T have a unique fixed point? Justify your answers.

3F Metric and Topological Spaces

Define the notion of a *connected component* of a space X.

If $A_\alpha \subset X$ are connected subsets of X such that $\bigcap_\alpha A_\alpha \neq \emptyset$, show that $\bigcup_\alpha A_\alpha$ is connected.

Prove that any point $x \in X$ is contained in a unique connected component.

Let $X \subset \mathbb{R}$ consist of the points $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$. What are the connected components of X?

4A Complex Methods

State the formula for the Laplace transform of a function $f(t)$, defined for $t \geqslant 0$.

Let $f(t)$ be periodic with period T (i.e. $f(t+T) = f(t)$). If $g(t)$ is defined to be equal to $f(t)$ in $[0,T]$ and zero elsewhere and its Laplace transform is $G(s)$, show that the Laplace transform of $f(t)$ is given by

$$F(s) \quad = \quad \frac{G(s)}{1 - e^{-sT}}\,.$$

Hence, or otherwise, find the inverse Laplace transform of

$$F(s) \quad = \quad \frac{1}{s}\frac{1 - e^{-sT/2}}{1 - e^{-sT}}\,.$$

5G Geometry

State a formula for the area of a hyperbolic triangle.

Hence, or otherwise, prove that if l_1 and l_2 are disjoint geodesics in the hyperbolic plane, there is at most one geodesic which is perpendicular to both l_1 and l_2.

6B Variational Principles

For a particle of unit mass moving freely on a unit sphere, the Lagrangian in polar coordinates is

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\sin^2\theta\,\dot{\phi}^2\,.$$

Find the equations of motion. Show that $l = \sin^2\theta\,\dot{\phi}$ is a conserved quantity, and use this result to simplify the equation of motion for θ. Deduce that

$$h = \dot{\theta}^2 + \frac{l^2}{\sin^2\theta}$$

is a conserved quantity. What is the interpretation of h?

7D Methods

For the step-function

$$F(x) = \begin{cases} 1\,, & |x| \leqslant 1/2 \\ 0\,, & \text{otherwise,} \end{cases}$$

its convolution with itself is the hat-function

$$G(x) = [F * F](x) = \begin{cases} 1 - |x|\,, & |x| \leqslant 1 \\ 0\,, & \text{otherwise.} \end{cases}$$

Find the Fourier transforms of F and G, and hence find the values of the integrals

$$I_1 = \int_{-\infty}^{\infty} \frac{\sin^2 y}{y^2}\,dy\,, \qquad I_2 = \int_{-\infty}^{\infty} \frac{\sin^4 y}{y^4}\,dy\,.$$

8C Quantum Mechanics

A one-dimensional quantum mechanical particle has normalised bound state energy eigenfunctions $\chi_n(x)$ and corresponding non-degenerate energy eigenvalues E_n. At $t = 0$ the normalised wavefunction $\psi(x, t)$ is given by

$$\psi(x, 0) = \sqrt{\frac{5}{6}}\, e^{ik_1}\chi_1(x) + \sqrt{\frac{1}{6}}\, e^{ik_2}\chi_2(x)$$

where k_1 and k_2 are real constants. Write down the expression for $\psi(x, t)$ at a later time t and give the probability that a measurement of the particle's energy will yield a value of E_2.

Show that the expectation value of x at time t is given by

$$\langle x \rangle = \frac{5}{6}\langle x \rangle_{11} + \frac{1}{6}\langle x \rangle_{22} + \frac{\sqrt{5}}{3}\mathrm{Re}\Big[\langle x \rangle_{12}\, e^{i(k_2-k_1)-i(E_2-E_1)t/\hbar}\Big]$$

where $\langle x \rangle_{ij} = \int_{-\infty}^{\infty} \chi_i^*(x)\, x\, \chi_j(x)\, dx$.

9H Markov Chains

A runner owns k pairs of running shoes and runs twice a day. In the morning she leaves her house by the front door, and in the evening she leaves by the back door. On starting each run she looks for shoes by the door through which she exits, and runs barefoot if none are there. At the end of each run she is equally likely to return through the front or back doors. She removes her shoes (if any) and places them by the door. In the morning of day 1 all shoes are by the back door so she must run barefoot.

Let $p_{00}^{(n)}$ be the probability that she runs barefoot on the morning of day $n + 1$. What conditions are satisfied in this problem which ensure $\lim_{n\to\infty} p_{00}^{(n)}$ exists? Show that its value is $1/2k$.

Find the expected number of days that will pass until the first morning that she finds all k pairs of shoes at her front door.

SECTION II

10F Linear Algebra

What is meant by the Jordan normal form of an $n \times n$ complex matrix?

Find the Jordan normal forms of the following matrices:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 9 & 6 & 3 & 0 \\ 15 & 12 & 9 & 3 \end{pmatrix}.$$

Suppose A is an invertible $n \times n$ complex matrix. Explain how to derive the characteristic and minimal polynomials of A^n from the characteristic and minimal polynomials of A. Justify your answer. [*Hint: write each polynomial as a product of linear factors.*]

11G Groups, Rings and Modules

For each of the following assertions, provide either a proof or a counterexample as appropriate:

(i) The ring $\mathbb{Z}_2[X]/\langle X^2+X+1\rangle$ is a field.

(ii) The ring $\mathbb{Z}_3[X]/\langle X^2+X+1\rangle$ is a field.

(iii) If F is a finite field, the ring $F[X]$ contains irreducible polynomials of arbitrarily large degree.

(iv) If R is the ring $C[0,1]$ of continuous real-valued functions on the interval $[0,1]$, and the non-zero elements $f, g \in R$ satisfy $f \mid g$ and $g \mid f$, then there is some unit $u \in R$ with $f = u \cdot g$.

12E Analysis II

Let f_n be a sequence of continuous functions on the interval $[0,1]$ such that $f_n(x) \to f(x)$ for each x. For the three statements:

(a) $f_n \to f$ uniformly on $[0,1]$;

(b) f is a continuous function;

(c) $\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$ as $n \to \infty$;

say which of the six possible implications $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$, $(b) \Rightarrow (a)$, $(b) \Rightarrow (c)$, $(c) \Rightarrow (a)$, $(c) \Rightarrow (b)$ are true and which false, giving in each case a proof or counterexample.

13E Complex Analysis

Let $D(a,R)$ denote the disc $|z-a|<R$ and let $f : D(a,R) \to \mathbb{C}$ be a holomorphic function. Using Cauchy's integral formula show that for every $r \in (0,R)$

$$f(a) = \int_0^1 f(a + re^{2\pi i t})\, dt\,.$$

Deduce that if for every $z \in D(a,R)$, $|f(z)| \leqslant |f(a)|$, then f is constant.

Let $f : D(0,1) \to D(0,1)$ be holomorphic with $f(0) = 0$. Show that $|f(z)| \leqslant |z|$ for all $z \in D(0,1)$. Moreover, show that if $|f(w)| = |w|$ for some $w \neq 0$, then there exists λ with $|\lambda| = 1$ such that $f(z) = \lambda z$ for all $z \in D(0,1)$.

14G Geometry

Define the *first* and *second fundamental forms* of a smooth surface $\Sigma \subset \mathbb{R}^3$, and explain their geometrical significance.

Write down the geodesic equations for a smooth curve $\gamma : [0,1] \to \Sigma$. Prove that γ is a geodesic if and only if the derivative of the tangent vector to γ is always orthogonal to Σ.

A plane $\Pi \subset \mathbb{R}^3$ cuts Σ in a smooth curve $C \subset \Sigma$, in such a way that reflection in the plane Π is an isometry of Σ (in particular, preserves Σ). Prove that C is a geodesic.

15D Methods

Consider Legendre's equation

$$(1-x^2)y'' - 2xy' + \lambda y = 0\,.$$

Show that if $\lambda = n(n+1)$, with n a non-negative integer, this equation has a solution $y = P_n(x)$, a polynomial of degree n. Find P_0, P_1 and P_2 explicitly, subject to the condition $P_n(1) = 1$.

The general solution of Laplace's equation $\nabla^2 \psi = 0$ in spherical polar coordinates, in the axisymmetric case, has the form

$$\psi(r,\theta) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-(n+1)}) P_n(\cos\theta)\,.$$

Hence, find the solution of Laplace's equation in the region $a \leqslant r \leqslant b$ satisfying the boundary conditions

$$\begin{cases} \psi(r,\theta) = 1\,, & r = a \\ \psi(r,\theta) = 3\cos^2\theta\,, & r = b\,. \end{cases}$$

16C Quantum Mechanics

State the condition for a linear operator $\hat{O}$ to be Hermitian.

Given the position and momentum operators $\hat{x}_i$ and $\hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}$, define the angular momentum operators $\hat{L}_i$. Establish the commutation relations

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k$$

and use these relations to show that $\hat{L}_3$ is Hermitian assuming $\hat{L}_1$ and $\hat{L}_2$ are.

Consider a wavefunction of the form

$$\chi(\mathbf{x}) = x_3(x_1 + kx_2)e^{-r}$$

where $r = |\mathbf{x}|$ and k is some constant. Show that $\chi(\mathbf{x})$ is an eigenstate of the total angular momentum operator $\hat{\mathbf{L}}^2$ for all k, and calculate the corresponding eigenvalue. For what values of k is $\chi(\mathbf{x})$ an eigenstate of $\hat{L}_3$? What are the corresponding eigenvalues?

17B Electromagnetism

Using the Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} - \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j},$$

show that in vacuum, $\mathbf{E}$ satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0, \qquad (*)$$

where $c^2 = (\epsilon_0 \mu_0)^{-1}$, as well as $\nabla \cdot \mathbf{E} = 0$. Also show that at a planar boundary between two media, $\mathbf{E}_t$ (the tangential component of $\mathbf{E}$) is continuous. Deduce that if one medium is of negligible resistance, $\mathbf{E}_t = 0$.

Consider an empty cubic box with walls of negligible resistance on the planes $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0$, $z = a$, where $a > 0$. Show that an electric field in the interior of the form

$$\begin{aligned} E_x &= f(x) \sin\left(\frac{m\pi y}{a}\right) \sin\left(\frac{n\pi z}{a}\right) e^{-i\omega t} \\ E_y &= g(y) \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{n\pi z}{a}\right) e^{-i\omega t} \\ E_z &= h(z) \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right) e^{-i\omega t}, \end{aligned}$$

with l, m and n positive integers, satisfies the boundary conditions on all six walls. Now suppose that

$$f(x) = f_0 \cos\left(\frac{l\pi x}{a}\right), \quad g(y) = g_0 \cos\left(\frac{m\pi y}{a}\right), \quad h(z) = h_0 \cos\left(\frac{n\pi z}{a}\right),$$

where f_0, g_0 and h_0 are constants. Show that the wave equation $(*)$ is satisfied, and determine the frequency ω. Find the further constraint on f_0, g_0 and h_0?

18A Fluid Dynamics

A rigid circular cylinder of radius a executes small amplitude oscillations with velocity $U(t)$ in a direction perpendicular to its axis, while immersed in an inviscid fluid of density ρ contained within a larger concentric fixed cylinder of radius b. Gravity is negligible. Neglecting terms quadratic in the amplitude, determine the boundary condition on the velocity on the inner cylinder, and calculate the velocity potential of the induced flow.

With the same approximations show that the difference in pressures on the surfaces of the two cylinders has magnitude

$$\rho \frac{\mathrm{d}U}{\mathrm{d}t} \frac{a(b-a)}{b+a} \cos\theta,$$

where θ is the azimuthal angle measured from the direction of U.

19D Numerical Analysis

Define the QR factorization of an $m \times n$ matrix A and explain how it can be used to solve the least squares problem of finding the vector $x^* \in \mathbb{R}^n$ which minimises $\|Ax^* - b\|$, where $b \in \mathbb{R}^m$, $m > n$, and the norm is the Euclidean one.

Define a Householder transformation H and show that it is an orthogonal matrix.

Using a Householder transformation, solve the least squares problem for

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 2 \end{bmatrix},$$

giving both x^* and $\|Ax^* - b\|$.

20H Statistics

Suppose that X is a single observation drawn from the uniform distribution on the interval $\left[\theta-10, \theta+10\right]$, where θ is unknown and might be any real number. Given $\theta_0 \neq 20$ we wish to test $H_0 : \theta = \theta_0$ against $H_1 : \theta = 20$. Let $\phi(\theta_0)$ be the test which accepts H_0 if and only if $X \in A(\theta_0)$, where

$$A(\theta_0) = \begin{cases} \left[\theta_0 - 8, \infty\right), & \theta_0 > 20 \\ \left(-\infty, \theta_0 + 8\right], & \theta_0 < 20 \,. \end{cases}$$

Show that this test has size $\alpha = 0.10$.

Now consider

$$\begin{aligned} C_1(X) &= \{\theta : X \in A(\theta)\}, \\ C_2(X) &= \left\{\theta : X - 9 \leqslant \theta \leqslant X + 9\right\}. \end{aligned}$$

Prove that both $C_1(X)$ and $C_2(X)$ specify 90% confidence intervals for θ. Find the confidence interval specified by $C_1(X)$ when $X = 0$.

Let $L_i(X)$ be the length of the confidence interval specified by $C_i(X)$. Let $\beta(\theta_0)$ be the probability of the Type II error of $\phi(\theta_0)$. Show that

$$E[L_1(X) \mid \theta = 20] = E\left[\int_{-\infty}^{\infty} 1_{\{\theta_0 \in C_1(X)\}} d\theta_0 \;\middle|\; \theta = 20\right] = \int_{-\infty}^{\infty} \beta(\theta_0)\, d\theta_0.$$

Here $1_{\{B\}}$ is an indicator variable for event B. The expectation is over X. [Orders of integration and expectation can be interchanged.]

Use what you know about constructing best tests to explain which of the two confidence intervals has the smaller expected length when $\theta = 20$.

21H Optimization

For given positive real numbers $(c_{ij} : \ i, j \in \{1, 2, 3\})$, consider the linear program

$$P: \text{ minimize} \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij} x_{ij},$$

$$\text{subject to } \sum_{i=1}^{3} x_{ij} \leqslant 1 \text{ for all } j, \quad \sum_{j=1}^{3} x_{ij} \geqslant 1 \text{ for all } i,$$

$$\text{and } x_{ij} \geqslant 0 \text{ for all } i, j.$$

(i) Consider the feasible solution x in which $x_{11} = x_{12} = x_{22} = x_{23} = x_{31} = x_{33} = 1/2$ and $x_{ij} = 0$ otherwise. Write down two basic feasible solutions of P, one of which you can be sure is at least as good as x. Are either of these basic feasible solutions of P degenerate?

(ii) Starting from a general definition of a Lagrangian dual problem show that the dual of P can be written as

$$D: \underset{\lambda_i \geqslant 0, \, \mu_i \geqslant 0}{\text{maximize}} \sum_{i=1}^{3} (\lambda_i - \mu_i) \quad \text{subject to } \lambda_i - \mu_j \leqslant c_{ij} \ \text{ for all } i, j.$$

What happens to the optimal value of this problem if the constraints $\lambda_i \geqslant 0$ and $\mu_i \geqslant 0$ are removed?

Prove that $x_{11} = x_{22} = x_{33} = 1$ is an optimal solution to P if and only if there exist $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_i - \lambda_j \leqslant c_{ij} - c_{jj}, \quad \text{for all } i, j.$$

[You may use any facts that you know from the general theory of linear programming provided that you state them.]

END OF PAPER