UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday, 8 June, 2011 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, ..., H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

Let V be an n-dimensional $\mathbb{R}$-vector space with an inner product. Let W be an m-dimensional subspace of V and $W^{\perp}$ its orthogonal complement, so that every element $v \in V$ can be uniquely written as $v = w + w'$ for $w \in W$ and $w' \in W^{\perp}$.

The *reflection map* with respect to W is defined as the linear map

$$f_W : V \ni w + w' \longmapsto w - w' \in V.$$

Show that f_W is an orthogonal transformation with respect to the inner product, and find its determinant.

2F Groups, Rings and Modules

Show that the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with $ij = k = -ji$, $i^2 = j^2 = k^2 = -1$, is not isomorphic to the symmetry group D_8 of the square.

3E Analysis II

Define differentiability of a function $f : \mathbb{R}^n \to \mathbb{R}$. Let $a > 0$ be a constant. For which points $(x, y) \in \mathbb{R}^2$ is

$$f(x, y) = |x|^a + |x - y|$$

differentiable? Justify your answer.

4G Metric and Topological Spaces

(i) Let $t > 0$. For $\mathbf{x} = (x, y)$, $\mathbf{x}' = (x', y') \in \mathbb{R}^2$, let

$$d(\mathbf{x}, \mathbf{x}') = |x' - x| + t|y' - y|,$$

$$\delta(\mathbf{x}, \mathbf{x}') = \sqrt{(x' - x)^2 + (y' - y)^2}.$$

(δ is the usual Euclidean metric on $\mathbb{R}^2$.) Show that d is a metric on $\mathbb{R}^2$ and that the two metrics d, δ give rise to the same topology on $\mathbb{R}^2$.

(ii) Give an example of a topology on $\mathbb{R}^2$, different from the one in (i), whose induced topology (subspace topology) on the x-axis is the usual topology (the one defined by the metric $d(x, x') = |x' - x|$). Justify your answer.

5A Methods

The Legendre equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

for $-1 \leqslant x \leqslant 1$ and non-negative integers n.

Write the Legendre equation as an eigenvalue equation for an operator L in Sturm-Liouville form. Show that L is self-adjoint and find the orthogonality relation between the eigenfunctions.

6C Electromagnetism

Maxwell's equations are

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}\,, \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\,,$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0\,, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}\,.$$

Find the equation relating ρ and $\mathbf{J}$ that must be satisfied for consistency, and give the interpretation of this equation.

Now consider the "magnetic limit" where $\rho = 0$ and the term $\epsilon_0\mu_0\dfrac{\partial\mathbf{E}}{\partial t}$ is neglected. Let $\mathbf{A}$ be a vector potential satisfying the gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, and assume the scalar potential vanishes. Find expressions for $\mathbf{E}$ and $\mathbf{B}$ in terms of $\mathbf{A}$ and show that Maxwell's equations are all satisfied provided $\mathbf{A}$ satisfies the appropriate Poisson equation.

7D Fluid Dynamics

A body of volume V lies totally submerged in a motionless fluid of uniform density ρ. Show that the force $\mathbf{F}$ on the body is given by

$$\mathbf{F} = -\int_S (p - p_0)\,\mathbf{n}\,dS$$

where p is the pressure in the fluid and p_0 is atmospheric pressure. You may use without proof the generalised divergence theorem in the form

$$\int_S \phi\,\mathbf{n}\,dS = \int_V \boldsymbol{\nabla}\phi\,dV.$$

Deduce that

$$\mathbf{F} = \rho g V \hat{\mathbf{z}},$$

where $\hat{\mathbf{z}}$ is the vertically upward unit vector. Interpret this result.

8H Statistics

Let $X_1, \ldots, X_n$ be random variables with joint density function $f(x_1, \ldots, x_n; \theta)$, where θ is an unknown parameter. The null hypothesis $H_0 : \theta = \theta_0$ is to be tested against the alternative hypothesis $H_1 : \theta = \theta_1$.

(i) Define the following terms: critical region, Type I error, Type II error, size, power.

(ii) State and prove the Neyman–Pearson lemma.

9H Optimization

Let $N = \{1, \ldots, n\}$ be the set of nodes of a network, where 1 is the source and n is the sink. Let c_{ij} denote the capacity of the arc from node i to node j.

(i) In the context of maximising the flow through this network, define the following terms: feasible flow, flow value, cut, cut capacity.

(ii) State and prove the max-flow min-cut theorem for network flows.

SECTION II

10G Linear Algebra

Let n be a positive integer, and let V be a $\mathbb{C}$-vector space of complex-valued functions on $\mathbb{R}$, generated by the set $\{\ \cos kx,\ \sin kx;\ k = 0, 1, \ldots, n-1\ \}$.

(i) Let $\langle f, g\rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx$ for $f, g \in V$. Show that this is a positive definite Hermitian form on V.

(ii) Let $\Delta(f) = \dfrac{d^2}{dx^2}f(x)$. Show that Δ is a self-adjoint linear transformation of V with respect to the form defined in (i).

(iii) Find an orthonormal basis of V with respect to the form defined in (i), which consists of eigenvectors of Δ.

11F Groups, Rings and Modules

Define the notion of a Euclidean domain and show that $\mathbb{Z}[i]$ is Euclidean.

Is $4 + i$ prime in $\mathbb{Z}[i]$?

12E Analysis II

What is meant by saying that two norms on a real vector space are Lipschitz equivalent?

Show that any two norms on $\mathbb{R}^n$ are Lipschitz equivalent. [You may assume that a continuous function on a closed bounded set in $\mathbb{R}^n$ has closed bounded image.]

Show that $\|f\|_1 = \int_{-1}^{1} |f(x)|\, dx$ defines a norm on the space $C[-1, 1]$ of continuous real-valued functions on $[-1, 1]$. Is it Lipschitz equivalent to the uniform norm? Justify your answer. Prove that the normed space $(C[-1, 1], \|\cdot\|_1)$ is not complete.

13A Complex Analysis or Complex Methods

(i) Let C be an anticlockwise contour defined by a square with vertices at $z = x+iy$ where

$$|x| = |y| = \left(2N + \frac{1}{2}\right)\pi,$$

for large integer N. Let

$$I = \oint_C \frac{\pi \cot z}{(z+\pi a)^4} dz.$$

Assuming that $I \to 0$ as $N \to \infty$, prove that, if a is not an integer, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^4} = \frac{\pi^4}{3\sin^2(\pi a)} \left(\frac{3}{\sin^2(\pi a)} - 2\right).$$

(ii) Deduce the value of

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^4}.$$

(iii) Briefly justify the assumption that $I \to 0$ as $N \to \infty$.

[*Hint: For part (iii) it is sufficient to consider, at most, one vertical side of the square and one horizontal side and to use a symmetry argument for the remaining sides.*]

14F Geometry

Suppose that $\pi : S^2 \to \mathbb{C}_\infty$ is stereographic projection. Show that, via π, every rotation of S^2 corresponds to a Möbius transformation in $PSU(2)$.

15D Variational Principles

(i) Let $I[y] = \displaystyle\int_0^1 \big((y')^2 - y^2\big)\,dx$, where y is twice differentiable and $y(0) = y(1) = 0$. Write down the associated Euler-Lagrange equation and show that the only solution is $y(x) = 0$.

(ii) Let $J[y] = \displaystyle\int_0^1 (y' + y\tan x)^2\,dx$, where y is twice differentiable and $y(0) = y(1) = 0$. Show that $J[y] = 0$ only if $y(x) = 0$.

(iii) Show that $I[y] = J[y]$ and deduce that the extremal value of $I[y]$ is a global minimum.

(iv) Use the second variation of $I[y]$ to verify that the extremal value of $I[y]$ is a local minimum.

(v) How would your answers to part (i) differ in the case $I[y] = \displaystyle\int_0^{2\pi} \big((y')^2 - y^2\big)\,dx$, where $y(0) = y(2\pi) = 0$? Show that the solution $y(x) = 0$ is not a global minimizer in this case. (You may use without proof the result $I[x(2\pi - x)] = -\frac{8}{15}(2\pi^2 - 5)$.) Explain why the arguments of parts (iii) and (iv) cannot be used.

16A Methods

Use a Green's function to find an integral expression for the solution of the equation

$$\frac{d^2\theta}{dt^2} + 4\frac{d\theta}{dt} + 29\,\theta = f(t)$$

for $t \geqslant 0$ subject to the initial conditions

$$\theta(0) = 0 \quad \text{and} \quad \frac{d\theta}{dt}(0) = 0.$$

17C Quantum Mechanics

The quantum mechanical angular momentum operators are

$$L_i = -i\hbar\,\epsilon_{ijk}\,x_j\,\frac{\partial}{\partial x_k} \qquad (i = 1, 2, 3).$$

Show that each of these is hermitian.

The total angular momentum operator is defined as $\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2$. Show that $\langle \mathbf{L}^2 \rangle \geqslant \langle L_3^2 \rangle$ in any state, and show that the only states where $\langle \mathbf{L}^2 \rangle = \langle L_3^2 \rangle$ are those with no angular dependence. Verify that the eigenvalues of the operators $\mathbf{L}^2$ and L_3^2 (whose values you may quote without proof) are consistent with these results.

18C Electromagnetism

(i) Consider an infinitely long solenoid parallel to the z-axis whose cross section is a simple closed curve of arbitrary shape. A current I, per unit length of the solenoid, flows around the solenoid parallel to the $x-y$ plane. Show using the relevant Maxwell equation that the magnetic field $\mathbf{B}$ inside the solenoid is uniform, and calculate its magnitude.

(ii) A wire loop in the shape of a regular hexagon of side length a carries a current I. Use the Biot-Savart law to calculate $\mathbf{B}$ at the centre of the loop.

19B Numerical Analysis

What is the QR-decomposition of a matrix A? Explain how to construct the matrices Q and R by the Gram-Schmidt procedure, and show how the decomposition can be used to solve the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ when A is a square matrix.

Why is this procedure not useful for numerical decomposition of large matrices? Give a brief description of an alternative procedure using Givens rotations.

Find a QR-decomposition for the matrix

$$\mathsf{A} = \begin{bmatrix} 3 & 4 & 7 & 13 \\ -6 & -8 & -8 & -12 \\ 3 & 4 & 7 & 11 \\ 0 & 2 & 5 & 7 \end{bmatrix}.$$

Is your decomposition unique? Use the decomposition you have found to solve the equation

$$\mathsf{A}\mathbf{x} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 9 \end{bmatrix}.$$

20H Markov Chains

(i) Let $(X_n)_{n\geqslant 0}$ be a Markov chain on the finite state space S with transition matrix P. Fix a subset $A \subseteq S$, and let

$$H = \inf\{n \geqslant 0 : X_n \in A\}.$$

Fix a function g on S such that $0 < g(i) \leqslant 1$ for all $i \in S$, and let

$$V_i = \mathbb{E}\left[\prod_{n=0}^{H-1} g(X_n) | X_0 = i\right]$$

where $\prod_{n=0}^{-1} a_n = 1$ by convention. Show that

$$V_i = \begin{cases} 1 & \text{if } i \in A \\ g(i)\sum_{j\in S} P_{ij}V_j & \text{otherwise.} \end{cases}$$

(ii) A flea lives on a polyhedron with N vertices, labelled $1, \ldots, N$. It hops from vertex to vertex in the following manner: if one day it is on vertex $i > 1$, the next day it hops to one of the vertices labelled $1, \ldots, i-1$ with equal probability, and it dies upon reaching vertex 1. Let X_n be the position of the flea on day n. What are the transition probabilities for the Markov chain $(X_n)_{n\geqslant 0}$?

(iii) Let H be the number of days the flea is alive, and let

$$V_i = \mathbb{E}(s^H | X_0 = i)$$

where s is a real number such that $0 < s \leqslant 1$. Show that $V_1 = 1$ and

$$\frac{i}{s}V_{i+1} = V_i + \frac{i-1}{s}V_i$$

for $i \geqslant 1$. Conclude that

$$\mathbb{E}(s^H | X_0 = N) = \prod_{i=1}^{N-1}\left(1 + \frac{s-1}{i}\right).$$

[Hint. Use part (i) with $A = \{1\}$ and a well-chosen function g.]

(iv) Show that

$$\mathbb{E}(H | X_0 = N) = \sum_{i=1}^{N-1}\frac{1}{i}.$$

END OF PAPER