

MATHEMATICAL TRIPOS Part IB

Tuesday, 7 June, 2011 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

*Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most **four** questions from Section I and at most **six** questions from Section II.*

Complete answers are preferred to fragments.

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in separate bundles labelled **A, B, ..., H** according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS

Gold cover sheets

Green master cover sheet

SPECIAL REQUIREMENTS

None

<p>You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.</p>

SECTION I**1G Linear Algebra**

(i) State the rank-nullity theorem for a linear map between finite-dimensional vector spaces.

(ii) Show that a linear transformation $f : V \rightarrow V$ of a finite-dimensional vector space V is bijective if it is injective or surjective.

(iii) Let V be the \mathbb{R} -vector space $\mathbb{R}[X]$ of all polynomials in X with coefficients in \mathbb{R} . Give an example of a linear transformation $f : V \rightarrow V$ which is surjective but not bijective.

2A Complex Analysis or Complex Methods

Derive the Cauchy-Riemann equations satisfied by the real and imaginary parts of a complex analytic function $f(z)$.

If $|f(z)|$ is constant on $|z| < 1$, prove that $f(z)$ is constant on $|z| < 1$.

3F Geometry

Suppose that $H \subseteq \mathbb{C}$ is the upper half-plane, $H = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$. Using the Riemannian metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$, define the length of a curve γ and the area of a region Ω in H .

Find the area of

$$\Omega = \{x + iy \mid |x| \leq \frac{1}{2}, x^2 + y^2 \geq 1\}.$$

4D Variational Principles

(i) Write down the Euler-Lagrange equations for the volume integral

$$\int_V (\nabla u \cdot \nabla u + 12u) dV,$$

where V is the unit ball $x^2 + y^2 + z^2 \leq 1$, and verify that the function $u(x, y, z) = x^2 + y^2 + z^2$ gives a stationary value of the integral subject to the condition $u = 1$ on the boundary.

(ii) Write down the Euler-Lagrange equations for the integral

$$\int_0^1 (\dot{x}^2 + \dot{y}^2 + 4x + 4y) dt,$$

where the dot denotes differentiation with respect to t , and verify that the functions $x(t) = t^2$, $y(t) = t^2$ give a stationary value of the integral subject to the boundary conditions $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$.

5B Fluid Dynamics

Inviscid fluid is contained in a square vessel with sides of length πL lying between $x = 0, \pi L$, $y = 0, \pi L$. The base of the container is at $z = -H$ where $H \gg L$ and the horizontal surface is at $z = 0$ when the fluid is at rest. The variation of pressure of the air above the fluid may be neglected.

Small amplitude surface waves are excited in the vessel.

(i) Now let $H \rightarrow \infty$. Explain why on dimensional grounds the frequencies ω of such waves are of the form

$$\omega = \left(\frac{\gamma g}{L} \right)^{\frac{1}{2}}$$

for some positive dimensionless constants γ , where g is the gravitational acceleration.

It is given that the velocity potential ϕ is of the form

$$\phi(x, y, z) \approx C \cos(mx/L) \cos(ny/L) e^{\gamma z/L}$$

where m and n are integers and C is a constant.

(ii) Why do cosines, rather than sines, appear in this expression?

(iii) Give an expression for γ in terms of m and n .

(iv) Give all possible values that γ^2 can take between 1 and 10 inclusive. How many different solutions for ϕ correspond to each of these values of γ^2 ?

6B Numerical Analysis

Orthogonal monic polynomials $p_0, p_1, \dots, p_n, \dots$ are defined with respect to the inner product $\langle p, q \rangle = \int_{-1}^1 w(x)p(x)q(x) dx$, where p_n is of degree n . Show that such polynomials obey a three-term recurrence relation

$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x)$$

for appropriate choices of α_n and β_n .

Now suppose that $w(x)$ is an even function of x . Show that the p_n are even or odd functions of x according to whether n is even or odd.

7H Statistics

Consider the experiment of tossing a coin n times. Assume that the tosses are independent and the coin is biased, with unknown probability p of heads and $1 - p$ of tails. A total of X heads is observed.

(i) What is the maximum likelihood estimator \hat{p} of p ?

Now suppose that a Bayesian statistician has the $\text{Beta}(M, N)$ prior distribution for p .

(ii) What is the posterior distribution for p ?

(iii) Assuming the loss function is $L(p, a) = (p - a)^2$, show that the statistician's point estimate for p is given by

$$\frac{M + X}{M + N + n}.$$

[The $\text{Beta}(M, N)$ distribution has density $\frac{\Gamma(M + N)}{\Gamma(M)\Gamma(N)}x^{M-1}(1 - x)^{N-1}$ for $0 < x < 1$ and mean $\frac{M}{M + N}$.]

8H Optimization

Suppose that $Ax \leq b$ and $x \geq 0$ and $A^T y \geq c$ and $y \geq 0$ where x and c are n -dimensional column vectors, y and b are m -dimensional column vectors, and A is an $m \times n$ matrix. Here, the vector inequalities are interpreted component-wise.

(i) Show that $c^T x \leq b^T y$.

(ii) Find the maximum value of

$$\begin{aligned} 6x_1 + 8x_2 + 3x_3 \quad \text{subject to} \quad & 2x_1 + 4x_2 + x_3 \leq 10, \\ & 3x_1 + 4x_2 + 3x_3 \leq 6, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

You should state any results from the course used in your solution.

SECTION II

9G Linear Algebra

Let V, W be finite-dimensional vector spaces over a field F and $f : V \rightarrow W$ a linear map.

(i) Show that f is injective if and only if the image of every linearly independent subset of V is linearly independent in W .

(ii) Define the dual space V^* of V and the dual map $f^* : W^* \rightarrow V^*$.

(iii) Show that f is surjective if and only if the image under f^* of every linearly independent subset of W^* is linearly independent in V^* .

10F Groups, Rings and Modules

(i) Suppose that G is a finite group of order $p^n r$, where p is prime and does not divide r . Prove the first Sylow theorem, that G has at least one subgroup of order p^n , and state the remaining Sylow theorems without proof.

(ii) Suppose that p, q are distinct primes. Show that there is no simple group of order pq .

11E Analysis II

What is meant by saying that a sequence of functions f_n converges uniformly to a function f ?

Let f_n be a sequence of differentiable functions on $[a, b]$ with f'_n continuous and such that $f_n(x_0)$ converges for some point $x_0 \in [a, b]$. Assume in addition that f'_n converges uniformly on $[a, b]$. Prove that f_n converges uniformly to a differentiable function f on $[a, b]$ and $f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$ for all $x \in [a, b]$. [You may assume that the uniform limit of continuous functions is continuous.]

Show that the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges for $s > 1$ and is uniformly convergent on $[1 + \varepsilon, \infty)$ for any $\varepsilon > 0$. Show that $\zeta(s)$ is differentiable on $(1, \infty)$ and

$$\zeta'(s) = - \sum_{n=2}^{\infty} \frac{\log n}{n^s}.$$

[You may use the Weierstrass M -test provided it is clearly stated.]

12G Metric and Topological Spaces

Let X be a metric space with the distance function $d : X \times X \rightarrow \mathbb{R}$. For a subset Y of X , its *diameter* is defined as $\delta(Y) := \sup\{d(y, y') \mid y, y' \in Y\}$.

Show that, if X is compact and $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open covering of X , then there exists an $\epsilon > 0$ such that every subset $Y \subset X$ with $\delta(Y) < \epsilon$ is contained in some U_λ .

13A Complex Analysis or Complex Methods

(i) Let $-1 < \alpha < 0$ and let

$$f(z) = \frac{\log(z - \alpha)}{z} \quad \text{where } -\pi \leq \arg(z - \alpha) < \pi,$$

$$g(z) = \frac{\log z}{z} \quad \text{where } -\pi \leq \arg(z) < \pi.$$

Here the logarithms take their principal values. Give a sketch to indicate the positions of the branch cuts implied by the definitions of $f(z)$ and $g(z)$.

(ii) Let $h(z) = f(z) - g(z)$. Explain why $h(z)$ is analytic in the annulus $1 \leq |z| \leq R$ for any $R > 1$. Obtain the first three terms of the Laurent expansion for $h(z)$ around $z = 0$ in this annulus and hence evaluate

$$\oint_{|z|=2} h(z) dz.$$

14A Methods

Let $f(t)$ be a real function defined on an interval $(-T, T)$ with Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right).$$

State and prove Parseval's theorem for $f(t)$ and its Fourier series. Write down the formulae for a_0 , a_n and b_n in terms of $f(t)$, $\cos \frac{n\pi t}{T}$ and $\sin \frac{n\pi t}{T}$.

Find the Fourier series of the square wave function defined on $(-\pi, \pi)$ by

$$g(t) = \begin{cases} 0 & -\pi < t \leq 0 \\ 1 & 0 < t < \pi. \end{cases}$$

Hence evaluate

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)}.$$

Using some of the above results evaluate

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

What is the sum of the Fourier series for $g(t)$ at $t = 0$? Comment on your answer.

15C Quantum Mechanics

For a quantum mechanical particle moving freely on a circle of length 2π , the wavefunction $\psi(t, x)$ satisfies the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

on the interval $0 \leq x \leq 2\pi$, and also the periodicity conditions $\psi(t, 2\pi) = \psi(t, 0)$, and $\frac{\partial \psi}{\partial x}(t, 2\pi) = \frac{\partial \psi}{\partial x}(t, 0)$. Find the allowed energy levels of the particle, and their degeneracies.

The current is defined as

$$j = \frac{i\hbar}{2m} \left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x} \right)$$

where ψ is a normalized state. Write down the general normalized state of the particle when it has energy $2\hbar^2/m$, and show that in any such state the current j is independent of x and t . Find a state with this energy for which the current has its maximum positive value, and find a state with this energy for which the current vanishes.

16D Electromagnetism

Starting from the relevant Maxwell equation, derive Gauss's law in integral form.

Use Gauss's law to obtain the potential at a distance r from an infinite straight wire with charge λ per unit length.

Write down the potential due to two infinite wires parallel to the z -axis, one at $x = y = 0$ with charge λ per unit length and the other at $x = 0, y = d$ with charge $-\lambda$ per unit length.

Find the potential and the electric field in the limit $d \rightarrow 0$ with $\lambda d = p$ where p is fixed. Sketch the equipotentials and the electric field lines.

17B Fluid Dynamics

A spherical bubble in an incompressible fluid of density ρ has radius $a(t)$. Write down an expression for the velocity field at a radius $R \geq a$.

The pressure far from the bubble is p_∞ . What is the pressure at radius R ?

Find conditions on a and its time derivatives that ensure that the maximum pressure in the fluid is reached at a radius R_{\max} where $a < R_{\max} < \infty$. Give an expression for this maximum pressure when the conditions hold.

Give the most general form of $a(t)$ that ensures that the pressure at $R = a(t)$ is p_∞ for all time.

18B Numerical Analysis

Consider a function $f(x)$ defined on the domain $x \in [0, 1]$. Find constants α, β, γ so that for any fixed $\xi \in [0, 1]$,

$$f''(\xi) = \alpha f(0) + \beta f'(0) + \gamma f(1)$$

is exactly satisfied for polynomials of degree less than or equal to two.

By using the Peano kernel theorem, or otherwise, show that

$$\begin{aligned} f'(\xi) - f'(0) - \xi(\alpha f(0) + \beta f'(0) + \gamma f(1)) &= \int_0^\xi (\xi - \theta) H_1(\theta) f'''(\theta) d\theta \\ &+ \int_0^\xi \theta H_2(\theta) f'''(\theta) d\theta + \int_\xi^1 \xi H_2(\theta) f'''(\theta) d\theta, \end{aligned}$$

where $H_1(\theta) = 1 - (1 - \theta)^2 \geq 0$, $H_2(\theta) = -(1 - \theta)^2 \leq 0$. Thus show that

$$|f'(\xi) - f'(0) - \xi(\alpha f(0) + \beta f'(0) + \gamma f(1))| \leq \frac{1}{6}(2\xi - 3\xi^2 + 4\xi^3 - \xi^4) \|f'''\|_\infty.$$

19H Statistics

Let X_1, \dots, X_n be independent random variables with probability mass function $f(x; \theta)$, where θ is an unknown parameter.

(i) What does it mean to say that T is a sufficient statistic for θ ? State, but do not prove, the factorisation criterion for sufficiency.

(ii) State and prove the Rao–Blackwell theorem.

Now consider the case where $f(x; \theta) = \frac{1}{x!}(-\log \theta)^x \theta$ for non-negative integer x and $0 < \theta < 1$.

(iii) Find a one-dimensional sufficient statistic T for θ .

(iv) Show that $\tilde{\theta} = \mathbb{1}_{\{X_1=0\}}$ is an unbiased estimator of θ .

(v) Find another unbiased estimator $\hat{\theta}$ which is a function of the sufficient statistic T and that has smaller variance than $\tilde{\theta}$. You may use the following fact without proof: $X_1 + \dots + X_n$ has the Poisson distribution with parameter $-n \log \theta$.

20H Markov Chains

Let $P = (p_{ij})_{i,j \in S}$ be the transition matrix for an irreducible Markov chain on the finite state space S .

- (i) What does it mean to say π is the invariant distribution for the chain?
- (ii) What does it mean to say the chain is in detailed balance with respect to π ?
- (iii) A symmetric random walk on a connected finite graph is the Markov chain whose state space is the set of vertices of the graph and whose transition probabilities are

$$p_{ij} = \begin{cases} 1/D_i & \text{if } j \text{ is adjacent to } i \\ 0 & \text{otherwise,} \end{cases}$$

where D_i is the number of vertices adjacent to vertex i . Show that the random walk is in detailed balance with respect to its invariant distribution.

- (iv) Let π be the invariant distribution for the transition matrix P , and define an inner product for vectors $x, y \in \mathbb{R}^S$ by the formula

$$\langle x, y \rangle = \sum_{i \in S} x_i \pi_i y_i.$$

Show that the equation

$$\langle x, Py \rangle = \langle Px, y \rangle$$

holds for all vectors $x, y \in \mathbb{R}^S$ if and only if the chain is in detailed balance with respect to π . [Here $z \in \mathbb{R}^S$ means $z = (z_i)_{i \in S}$.]

END OF PAPER