UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2011

List of Courses

Paper 2, Section I

3E Analysis II

Define differentiability of a function $f : \mathbb{R}^n \to \mathbb{R}$. Let $a > 0$ be a constant. For which points $(x, y) \in \mathbb{R}^2$ is

$$f(x, y) = |x|^a + |x - y|$$

differentiable? Justify your answer.

Paper 3, Section I

2E Analysis II

Suppose f is a uniformly continuous mapping from a metric space X to a metric space Y. Prove that $f(x_n)$ is a Cauchy sequence in Y for every Cauchy sequence x_n in X.

Let f be a continuous mapping between metric spaces and suppose that f has the property that $f(x_n)$ is a Cauchy sequence whenever x_n is a Cauchy sequence. Is it true that f must be uniformly continuous? Justify your answer.

Paper 4, Section I

3E Analysis II

Let $B[0, 1]$ denote the set of bounded real-valued functions on $[0, 1]$. A distance d on $B[0, 1]$ is defined by

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Given that $(B[0, 1], d)$ is a metric space, show that it is complete. Show that the subset $C[0, 1] \subset B[0, 1]$ of continuous functions is a closed set.

Paper 1, Section II

11E Analysis II

What is meant by saying that a sequence of functions f_n converges uniformly to a function f?

Let f_n be a sequence of differentiable functions on $[a, b]$ with f_n' continuous and such that $f_n(x_0)$ converges for some point $x_0 \in [a, b]$. Assume in addition that f_n' converges uniformly on $[a, b]$. Prove that f_n converges uniformly to a differentiable function f on $[a, b]$ and $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a, b]$. [You may assume that the uniform limit of continuous functions is continuous.]

Show that the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges for $s > 1$ and is uniformly convergent on $[1 + \varepsilon, \infty)$ for any $\varepsilon > 0$. Show that $\zeta(s)$ is differentiable on $(1, \infty)$ and

$$\zeta'(s) = -\sum_{n=2}^{\infty} \frac{\log n}{n^s}.$$

[You may use the Weierstrass M-test provided it is clearly stated.]

Paper 2, Section II

12E Analysis II

What is meant by saying that two norms on a real vector space are Lipschitz equivalent?

Show that any two norms on $\mathbb{R}^n$ are Lipschitz equivalent. [You may assume that a continuous function on a closed bounded set in $\mathbb{R}^n$ has closed bounded image.]

Show that $\|f\|_1 = \int_{-1}^{1} |f(x)|\, dx$ defines a norm on the space $C[-1, 1]$ of continuous real-valued functions on $[-1, 1]$. Is it Lipschitz equivalent to the uniform norm? Justify your answer. Prove that the normed space $(C[-1, 1], \|\cdot\|_1)$ is not complete.

Paper 3, Section II

12E Analysis II

Consider a map $f : \mathbb{R}^n \to \mathbb{R}^m$.

Assume f is differentiable at x and let $D_x f$ denote the derivative of f at x. Show that

$$D_x f(v) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

for any $v \in \mathbb{R}^n$.

Assume now that f is such that for some fixed x and for every $v \in \mathbb{R}^n$ the limit

$$\lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

exists. Is it true that f is differentiable at x? Justify your answer.

Let M_k denote the set of all $k \times k$ real matrices which is identified with $\mathbb{R}^{k^2}$. Consider the function $f : M_k \to M_k$ given by $f(A) = A^3$. Explain why f is differentiable. Show that the derivative of f at the matrix A is given by

$$D_A f(H) = HA^2 + AHA + A^2 H$$

for any matrix $H \in M_k$. State carefully the inverse function theorem and use it to prove that there exist open sets U and V containing the identity matrix such that given $B \in V$ there exists a unique $A \in U$ such that $A^3 = B$.

Paper 4, Section II

12E Analysis II

Define a contraction mapping and state the contraction mapping theorem.

Let (X, d) be a non-empty complete metric space and let $\phi : X \to X$ be a map. Set $\phi^1 = \phi$ and $\phi^{n+1} = \phi \circ \phi^n$. Assume that for some integer $r \geqslant 1$, ϕ^r is a contraction mapping. Show that ϕ has a unique fixed point y and that any $x \in X$ has the property that $\phi^n(x) \to y$ as $n \to \infty$.

Let $C[0,1]$ be the set of continuous real-valued functions on $[0,1]$ with the uniform norm. Suppose $T : C[0,1] \to C[0,1]$ is defined by

$$T(f)(x) = \int_0^x f(t)\,dt$$

for all $x \in [0,1]$ and $f \in C[0,1]$. Show that T is not a contraction mapping but that T^2 is.

Paper 4, Section I

4E Complex Analysis

Let $f(z)$ be an analytic function in an open subset U of the complex plane. Prove that f has derivatives of all orders at any point z in U. [You may assume Cauchy's integral formula provided it is clearly stated.]

Paper 3, Section II

13E Complex Analysis

Let $g : \mathbb{C} \to \mathbb{C}$ be a continuous function such that

$$\int_\Gamma g(z)\, dz = 0$$

for any closed curve Γ which is the boundary of a rectangle in $\mathbb{C}$ with sides parallel to the real and imaginary axes. Prove that g is analytic.

Let $f : \mathbb{C} \to \mathbb{C}$ be continuous. Suppose in addition that f is analytic at every point $z \in \mathbb{C}$ with non-zero imaginary part. Show that f is analytic at every point in $\mathbb{C}$.

Let $\mathbb{H}$ be the upper half-plane of complex numbers z with positive imaginary part $\Im(z) > 0$. Consider a continuous function $F : \mathbb{H} \cup \mathbb{R} \to \mathbb{C}$ such that F is analytic on $\mathbb{H}$ and $F(\mathbb{R}) \subset \mathbb{R}$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \begin{cases} F(z) & \text{if } \Im(z) \geqslant 0 \\ \\ \overline{F(\overline{z})} & \text{if } \Im(z) \leqslant 0. \end{cases}$$

Show that f is analytic.

Paper 1, Section I

2A Complex Analysis or Complex Methods

Derive the Cauchy-Riemann equations satisfied by the real and imaginary parts of a complex analytic function $f(z)$.

If $|f(z)|$ is constant on $|z| < 1$, prove that $f(z)$ is constant on $|z| < 1$.

Paper 1, Section II

13A Complex Analysis or Complex Methods

(i) Let $-1 < \alpha < 0$ and let

$$\begin{aligned} f(z) &= \frac{\log(z-\alpha)}{z} \quad \text{where } -\pi \leqslant \arg(z-\alpha) < \pi\,, \\ g(z) &= \frac{\log z}{z} \qquad\quad \text{where } -\pi \leqslant \arg(z) < \pi\,. \end{aligned}$$

Here the logarithms take their principal values. Give a sketch to indicate the positions of the branch cuts implied by the definitions of $f(z)$ and $g(z)$.

(ii) Let $h(z) = f(z) - g(z)$. Explain why $h(z)$ is analytic in the annulus $1 \leqslant |z| \leqslant R$ for any $R > 1$. Obtain the first three terms of the Laurent expansion for $h(z)$ around $z = 0$ in this annulus and hence evaluate

$$\oint_{|z|=2} h(z)dz\,.$$

Paper 2, Section II

13A Complex Analysis or Complex Methods

(i) Let C be an anticlockwise contour defined by a square with vertices at $z = x + iy$ where

$$|x| = |y| = \left(2N + \frac{1}{2}\right)\pi,$$

for large integer N. Let

$$I = \oint_C \frac{\pi \cot z}{(z + \pi a)^4} dz.$$

Assuming that $I \to 0$ as $N \to \infty$, prove that, if a is not an integer, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^4} = \frac{\pi^4}{3\sin^2(\pi a)}\left(\frac{3}{\sin^2(\pi a)} - 2\right).$$

(ii) Deduce the value of

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^4}.$$

(iii) Briefly justify the assumption that $I \to 0$ as $N \to \infty$.

[*Hint: For part (iii) it is sufficient to consider, at most, one vertical side of the square and one horizontal side and to use a symmetry argument for the remaining sides.*]

Paper 3, Section I

4D Complex Methods

Write down the function $\psi(u,v)$ that satisfies

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}=0\,,\qquad \psi(-\tfrac{1}{2},v)=-1,\quad \psi(\tfrac{1}{2},v)=1\,.$$

The circular arcs $\mathcal{C}_1$ and $\mathcal{C}_2$ in the complex z-plane are defined by

$$|z+1|=1,\ z\neq 0\ \text{ and }\ |z-1|=1,\ z\neq 0\,,$$

respectively. You may assume without proof that the mapping from the complex z-plane to the complex ζ-plane defined by

$$\zeta=\frac{1}{z}$$

takes $\mathcal{C}_1$ to the line $u=-\frac{1}{2}$ and $\mathcal{C}_2$ to the line $u=\frac{1}{2}$, where $\zeta=u+iv$, and that the region $\mathcal{D}$ in the z-plane exterior to both the circles $|z+1|=1$ and $|z-1|=1$ maps to the region in the ζ-plane given by $-\frac{1}{2}<u<\frac{1}{2}$.

Use the above mapping to solve the problem

$$\nabla^2\phi=0\quad\text{ in }\mathcal{D},\qquad \phi=-1\text{ on }\mathcal{C}_1\text{ and }\phi=1\text{ on }\mathcal{C}_2.$$

Paper 4, Section II

14D Complex Methods

State and prove the convolution theorem for Laplace transforms.

Use Laplace transforms to solve

$$2f'(t)-\int_0^t(t-\tau)^2f(\tau)\,d\tau=4tH(t)$$

with $f(0)=0$, where $H(t)$ is the Heaviside function. You may assume that the Laplace transform, $\widehat{f}(s)$, of $f(t)$ exists for $\operatorname{Re}s$ sufficiently large.

Paper 2, Section I

6C Electromagnetism

Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Find the equation relating ρ and $\mathbf{J}$ that must be satisfied for consistency, and give the interpretation of this equation.

Now consider the "magnetic limit" where $\rho = 0$ and the term $\epsilon_0 \mu_0 \dfrac{\partial \mathbf{E}}{\partial t}$ is neglected. Let $\mathbf{A}$ be a vector potential satisfying the gauge condition $\nabla \cdot \mathbf{A} = 0$, and assume the scalar potential vanishes. Find expressions for $\mathbf{E}$ and $\mathbf{B}$ in terms of $\mathbf{A}$ and show that Maxwell's equations are all satisfied provided $\mathbf{A}$ satisfies the appropriate Poisson equation.

Paper 4, Section I

7C Electromagnetism

A plane electromagnetic wave in a vacuum has electric field

$$\mathbf{E} = (E_0 \sin k(z - ct), 0, 0).$$

What are the wavevector, polarization vector and speed of the wave? Using Maxwell's equations, find the magnetic field $\mathbf{B}$. Assuming the scalar potential vanishes, find a possible vector potential $\mathbf{A}$ for this wave, and verify that it gives the correct $\mathbf{E}$ and $\mathbf{B}$.

Paper 1, Section II

16D Electromagnetism

Starting from the relevant Maxwell equation, derive Gauss's law in integral form.

Use Gauss's law to obtain the potential at a distance r from an infinite straight wire with charge λ per unit length.

Write down the potential due to two infinite wires parallel to the z-axis, one at $x = y = 0$ with charge λ per unit length and the other at $x = 0$, $y = d$ with charge $-\lambda$ per unit length.

Find the potential and the electric field in the limit $d \to 0$ with $\lambda d = p$ where p is fixed. Sketch the equipotentials and the electric field lines.

Paper 2, Section II

18C Electromagnetism

(i) Consider an infinitely long solenoid parallel to the z-axis whose cross section is a simple closed curve of arbitrary shape. A current I, per unit length of the solenoid, flows around the solenoid parallel to the $x-y$ plane. Show using the relevant Maxwell equation that the magnetic field $\mathbf{B}$ inside the solenoid is uniform, and calculate its magnitude.

(ii) A wire loop in the shape of a regular hexagon of side length a carries a current I. Use the Biot-Savart law to calculate $\mathbf{B}$ at the centre of the loop.

Paper 3, Section II

17C Electromagnetism

Show, using the vacuum Maxwell equations, that for any volume V with surface S,

$$\frac{d}{dt}\int_V \left(\frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E}+\frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}\right)dV=\int_S\left(-\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}\right)\cdot\mathbf{dS}\,.$$

What is the interpretation of this equation?

A uniform straight wire, with a circular cross section of radius r, has conductivity σ and carries a current I. Calculate $\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ at the surface of the wire, and hence find the flux of $\frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ into unit length of the wire. Relate your result to the resistance of the wire, and the rate of energy dissipation.

Paper 1, Section I

5B Fluid Dynamics

Inviscid fluid is contained in a square vessel with sides of length πL lying between $x = 0, \pi L$, $y = 0, \pi L$. The base of the container is at $z = -H$ where $H \gg L$ and the horizontal surface is at $z = 0$ when the fluid is at rest. The variation of pressure of the air above the fluid may be neglected.

Small amplitude surface waves are excited in the vessel.

(i) Now let $H \to \infty$. Explain why on dimensional grounds the frequencies ω of such waves are of the form

$$\omega = \left(\frac{\gamma g}{L}\right)^{\frac{1}{2}}$$

for some positive dimensionless constants γ, where g is the gravitational acceleration.

It is given that the velocity potential ϕ is of the form

$$\phi(x, y, z) \approx C \cos(mx/L) \cos(ny/L) \mathrm{e}^{\gamma z/L}$$

where m and n are integers and C is a constant.

(ii) Why do cosines, rather than sines, appear in this expression?

(iii) Give an expression for γ in terms of m and n.

(iv) Give all possible values that γ^2 can take between 1 and 10 inclusive. How many different solutions for ϕ correspond to each of these values of γ^2?

Paper 2, Section I

7D Fluid Dynamics

A body of volume V lies totally submerged in a motionless fluid of uniform density ρ. Show that the force $\mathbf{F}$ on the body is given by

$$\mathbf{F} = -\int_S (p - p_0)\, \mathbf{n}\, dS$$

where p is the pressure in the fluid and p_0 is atmospheric pressure. You may use without proof the generalised divergence theorem in the form

$$\int_S \phi\, \mathbf{n}\, dS = \int_V \boldsymbol{\nabla} \phi\, dV.$$

Deduce that

$$\mathbf{F} = \rho g V \hat{\mathbf{z}},$$

where $\hat{\mathbf{z}}$ is the vertically upward unit vector. Interpret this result.

Paper 1, Section II
17B Fluid Dynamics

A spherical bubble in an incompressible fluid of density ρ has radius $a(t)$. Write down an expression for the velocity field at a radius $R \geqslant a$.

The pressure far from the bubble is p_∞. What is the pressure at radius R?

Find conditions on a and its time derivatives that ensure that the maximum pressure in the fluid is reached at a radius R_{max} where $a < R_{\text{max}} < \infty$. Give an expression for this maximum pressure when the conditions hold.

Give the most general form of $a(t)$ that ensures that the pressure at $R = a(t)$ is p_∞ for all time.

Paper 3, Section II
18D Fluid Dynamics

Water of constant density ρ flows steadily through a long cylindrical tube, the wall of which is elastic. The exterior radius of the tube at a distance z along the tube, $r(z)$, is determined by the pressure in the tube, $p(z)$, according to

$$r(z) = r_0 + b(p(z) - p_0),$$

where r_0 and p_0 are the radius and pressure far upstream ($z \to -\infty$), and b is a positive constant.

The interior radius of the tube is $r(z) - h(z)$, where $h(z)$, the thickness of the wall, is a given slowly varying function of z which is zero at both ends of the pipe. The velocity of the water in the pipe is $u(z)$ and the water enters the pipe at velocity V.

Show that $u(z)$ satisfies

$$H = 1 - v^{-\frac{1}{2}} + \tfrac{1}{4}k(1 - v^2)\,,$$

where $H = \dfrac{h}{r_0}$, $v = \dfrac{u}{V}$ and $k = \dfrac{2b\rho V^2}{r_0}$. Sketch the graph of H against v.

Let H_m be the maximum value of H in the tube. Show that the flow is only possible if H_m does not exceed a certain critical value H_c. Find H_c in terms of k.

Show that, under conditions to be determined (which include a condition on the value of k), the water can leave the pipe with speed less than V.

Paper 4, Section II

18D Fluid Dynamics

Show that an irrotational incompressible flow can be determined from a velocity potential ϕ that satisfies $\nabla^2\phi = 0$.

Given that the general solution of $\nabla^2\phi = 0$ in plane polar coordinates is

$$\phi = \sum_{n=-\infty}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n + c \log r + b\theta \,,$$

obtain the corresponding fluid velocity.

A two-dimensional irrotational incompressible fluid flows past the circular disc with boundary $r = a$. For large r, the flow is uniform and parallel to the x-axis ($x = r\cos\theta$). Write down the boundary conditions for large r and on $r = a$, and hence derive the velocity potential in the form

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{\kappa\theta}{2\pi}\,,$$

where κ is the circulation.

Show that the acceleration of the fluid at $r = a$ and $\theta = 0$ is

$$\frac{\kappa}{2\pi a^2}\left(-\frac{\kappa}{2\pi a}\mathbf{e}_r - 2U\mathbf{e}_\theta\right).$$

Paper 1, Section I

3F Geometry

Suppose that $H \subseteq \mathbb{C}$ is the upper half-plane, $H = \{x + iy \mid x, y \in \mathbb{R},\ y > 0\}$. Using the Riemannian metric $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$, define the length of a curve γ and the area of a region Ω in H.

Find the area of

$$\Omega = \{x + iy \mid |x| \leqslant \frac{1}{2},\ x^2 + y^2 \geqslant 1\}\,.$$

Paper 3, Section I

5F Geometry

Let $R(x, \theta)$ denote anti-clockwise rotation of the Euclidean plane $\mathbb{R}^2$ through an angle θ about a point x.

Show that $R(x, \theta)$ is a composite of two reflexions.

Assume $\theta, \phi \in (0, \pi)$. Show that the composite $R(y, \phi) \cdot R(x, \theta)$ is also a rotation $R(z, \psi)$. Find z and ψ.

Paper 2, Section II

14F Geometry

Suppose that $\pi : S^2 \to \mathbb{C}_\infty$ is stereographic projection. Show that, via π, every rotation of S^2 corresponds to a Möbius transformation in $PSU(2)$.

Paper 3, Section II

14F Geometry

Suppose that $\eta(u) = (f(u), 0, g(u))$ is a unit speed curve in $\mathbb{R}^3$. Show that the corresponding surface of revolution S obtained by rotating this curve about the z-axis has Gaussian curvature $K = -(d^2 f/du^2)/f$.

Paper 4, Section II

15F Geometry

Suppose that P is a point on a Riemannian surface S. Explain the notion of geodesic polar co-ordinates on S in a neighbourhood of P, and prove that if C is a geodesic circle centred at P of small positive radius, then the geodesics through P meet C at right angles.

Paper 2, Section I

2F Groups, Rings and Modules

Show that the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with $ij = k = -ji$, $i^2 = j^2 = k^2 = -1$, is not isomorphic to the symmetry group D_8 of the square.

Paper 3, Section I

1F Groups, Rings and Modules

Suppose that A is an integral domain containing a field K and that A is finite-dimensional as a K-vector space. Prove that A is a field.

Paper 4, Section I

2F Groups, Rings and Modules

A ring R satisfies the descending chain condition (DCC) on ideals if, for every sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ of ideals in R, there exists n with $I_n = I_{n+1} = I_{n+2} = \ldots$. Show that $\mathbb{Z}$ does not satisfy the DCC on ideals.

Paper 1, Section II

10F Groups, Rings and Modules

(i) Suppose that G is a finite group of order $p^n r$, where p is prime and does not divide r. Prove the first Sylow theorem, that G has at least one subgroup of order p^n, and state the remaining Sylow theorems without proof.

(ii) Suppose that p, q are distinct primes. Show that there is no simple group of order pq.

Paper 2, Section II

11F Groups, Rings and Modules

Define the notion of a Euclidean domain and show that $\mathbb{Z}[i]$ is Euclidean.

Is $4 + i$ prime in $\mathbb{Z}[i]$?

Paper 3, Section II

11F Groups, Rings and Modules

Suppose that A is a matrix over $\mathbb{Z}$. What does it mean to say that A can be brought to Smith normal form?

Show that the structure theorem for finitely generated modules over $\mathbb{Z}$ (which you should state) follows from the existence of Smith normal forms for matrices over $\mathbb{Z}$.

Bring the matrix $\begin{pmatrix} -4 & -6 \\ 2 & 2 \end{pmatrix}$ to Smith normal form.

Suppose that M is the $\mathbb{Z}$-module with generators e_1, e_2, subject to the relations

$$-4e_1 + 2e_2 = -6e_1 + 2e_2 = 0\,.$$

Describe M in terms of the structure theorem.

Paper 4, Section II

11F Groups, Rings and Modules

State and prove the Hilbert Basis Theorem.

Is every ring Noetherian? Justify your answer.

Paper 1, Section I

1G Linear Algebra

(i) State the rank-nullity theorem for a linear map between finite-dimensional vector spaces.

(ii) Show that a linear transformation $f : V \to V$ of a finite-dimensional vector space V is bijective if it is injective or surjective.

(iii) Let V be the $\mathbb{R}$-vector space $\mathbb{R}[X]$ of all polynomials in X with coefficients in $\mathbb{R}$. Give an example of a linear transformation $f : V \to V$ which is surjective but not bijective.

Paper 2, Section I

1G Linear Algebra

Let V be an n-dimensional $\mathbb{R}$-vector space with an inner product. Let W be an m-dimensional subspace of V and $W^{\perp}$ its orthogonal complement, so that every element $v \in V$ can be uniquely written as $v = w + w'$ for $w \in W$ and $w' \in W^{\perp}$.

The *reflection map* with respect to W is defined as the linear map

$$f_W : V \ni w + w' \longmapsto w - w' \in V.$$

Show that f_W is an orthogonal transformation with respect to the inner product, and find its determinant.

Paper 4, Section I

1G Linear Algebra

(i) Let V be a vector space over a field F, and W_1, W_2 subspaces of V. Define the subset $W_1 + W_2$ of V, and show that $W_1 + W_2$ and $W_1 \cap W_2$ are subspaces of V.

(ii) When W_1, W_2 are finite-dimensional, state a formula for $\dim(W_1 + W_2)$ in terms of $\dim W_1$, $\dim W_2$ and $\dim(W_1 \cap W_2)$.

(iii) Let V be the $\mathbb{R}$-vector space of all $n \times n$ matrices over $\mathbb{R}$. Let S be the subspace of all symmetric matrices and T the subspace of all upper triangular matrices (the matrices (a_{ij}) such that $a_{ij} = 0$ whenever $i > j$). Find $\dim S$, $\dim T$, $\dim(S \cap T)$ and $\dim(S + T)$. Briefly justify your answer.

Paper 1, Section II

9G Linear Algebra

Let V, W be finite-dimensional vector spaces over a field F and $f : V \to W$ a linear map.

(i) Show that f is injective if and only if the image of every linearly independent subset of V is linearly independent in W.

(ii) Define the dual space V^* of V and the dual map $f^* : W^* \to V^*$.

(iii) Show that f is surjective if and only if the image under f^* of every linearly independent subset of W^* is linearly independent in V^*.

Paper 2, Section II

10G Linear Algebra

Let n be a positive integer, and let V be a $\mathbb{C}$-vector space of complex-valued functions on $\mathbb{R}$, generated by the set $\{ \cos kx, \ \sin kx; \ k = 0, 1, \ldots, n-1 \}$.

(i) Let $\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx$ for $f, g \in V$. Show that this is a positive definite Hermitian form on V.

(ii) Let $\Delta(f) = \dfrac{d^2}{dx^2} f(x)$. Show that Δ is a self-adjoint linear transformation of V with respect to the form defined in (i).

(iii) Find an orthonormal basis of V with respect to the form defined in (i), which consists of eigenvectors of Δ.

Paper 3, Section II

10G Linear Algebra

(i) Let A be an $n \times n$ complex matrix and $f(X)$ a polynomial with complex coefficients. By considering the Jordan normal form of A or otherwise, show that if the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ then the eigenvalues of $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$.

(ii) Let $B = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix}$. Write B as $B = f(A)$ for a polynomial f with $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and find the eigenvalues of B.

[Hint: compute the powers of A.]

Paper 4, Section II

10G Linear Algebra

Let V be an n-dimensional $\mathbb{R}$-vector space and $f, g : V \to V$ linear transformations. Suppose f is invertible and diagonalisable, and $f \circ g = t \cdot (g \circ f)$ for some real number $t > 1$.

(i) Show that g is nilpotent, i.e. some positive power of g is 0.

(ii) Suppose that there is a non-zero vector $v \in V$ with $f(v) = v$ and $g^{n-1}(v) \neq 0$. Determine the diagonal form of f.

Paper 3, Section I

9H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain with state space S.

(i) What does it mean to say that $(X_n)_{n\geqslant 0}$ has the strong Markov property? Your answer should include the definition of the term *stopping time.*

(ii) Show that

$$\mathbb{P}(X_n = i \text{ at least } k \text{ times } |X_0 = i) = \big[\mathbb{P}(X_n = i \text{ at least once } |X_0 = i)\big]^k$$

for a state $i \in S$. You may use without proof the fact that $(X_n)_{n\geqslant 0}$ has the strong Markov property.

Paper 4, Section I

9H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain on a state space S, and let $p_{ij}(n) = \mathbb{P}(X_n = j|X_0 = i)$.

(i) What does the term *communicating class* mean in terms of this chain?

(ii) Show that $p_{ii}(m+n) \geqslant p_{ij}(m)p_{ji}(n)$.

(iii) The period d_i of a state i is defined to be

$$d_i = \gcd\{n \geqslant 1 : p_{ii}(n) > 0\}.$$

Show that if i and j are in the same communicating class and $p_{jj}(r) > 0$, then d_i divides r.

Paper 1, Section II

20H Markov Chains

Let $P = (p_{ij})_{i,j \in S}$ be the transition matrix for an irreducible Markov chain on the finite state space S.

(i) What does it mean to say π is the invariant distribution for the chain?

(ii) What does it mean to say the chain is in detailed balance with respect to π?

(iii) A symmetric random walk on a connected finite graph is the Markov chain whose state space is the set of vertices of the graph and whose transition probabilities are

$$p_{ij} = \begin{cases} 1/D_i & \text{if } j \text{ is adjacent to } i \\ 0 & \text{otherwise,} \end{cases}$$

where D_i is the number of vertices adjacent to vertex i. Show that the random walk is in detailed balance with respect to its invariant distribution.

(iv) Let π be the invariant distribution for the transition matrix P, and define an inner product for vectors $x, y \in \mathbb{R}^S$ by the formula

$$\langle x, y \rangle = \sum_{i \in S} x_i \pi_i y_i .$$

Show that the equation

$$\langle x, Py \rangle = \langle Px, y \rangle$$

holds for all vectors $x, y \in \mathbb{R}^S$ if and only if the chain is in detailed balance with respect to π. [Here $z \in \mathbb{R}^S$ means $z = (z_i)_{i \in S}$.]

Paper 2, Section II

20H Markov Chains

(i) Let $(X_n)_{n\geqslant 0}$ be a Markov chain on the finite state space S with transition matrix P. Fix a subset $A \subseteq S$, and let

$$H = \inf\{n \geqslant 0 : X_n \in A\}.$$

Fix a function g on S such that $0 < g(i) \leqslant 1$ for all $i \in S$, and let

$$V_i = \mathbb{E}\left[\prod_{n=0}^{H-1} g(X_n) | X_0 = i\right]$$

where $\prod_{n=0}^{-1} a_n = 1$ by convention. Show that

$$V_i = \begin{cases} 1 & \text{if } i \in A \\ g(i)\sum_{j\in S} P_{ij}V_j & \text{otherwise.} \end{cases}$$

(ii) A flea lives on a polyhedron with N vertices, labelled $1, \ldots, N$. It hops from vertex to vertex in the following manner: if one day it is on vertex $i > 1$, the next day it hops to one of the vertices labelled $1, \ldots, i-1$ with equal probability, and it dies upon reaching vertex 1. Let X_n be the position of the flea on day n. What are the transition probabilities for the Markov chain $(X_n)_{n\geqslant 0}$?

(iii) Let H be the number of days the flea is alive, and let

$$V_i = \mathbb{E}(s^H | X_0 = i)$$

where s is a real number such that $0 < s \leqslant 1$. Show that $V_1 = 1$ and

$$\frac{i}{s}V_{i+1} = V_i + \frac{i-1}{s}V_i$$

for $i \geqslant 1$. Conclude that

$$\mathbb{E}(s^H | X_0 = N) = \prod_{i=1}^{N-1}\left(1 + \frac{s-1}{i}\right).$$

[Hint. Use part (i) with $A = \{1\}$ and a well-chosen function g.]

(iv) Show that

$$\mathbb{E}(H | X_0 = N) = \sum_{i=1}^{N-1}\frac{1}{i}.$$

Paper 2, Section I

5A Methods

The Legendre equation is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

for $-1\leqslant x\leqslant 1$ and non-negative integers n.

Write the Legendre equation as an eigenvalue equation for an operator L in Sturm-Liouville form. Show that L is self-adjoint and find the orthogonality relation between the eigenfunctions.

Paper 3, Section I

7A Methods

The Fourier transform $\widetilde{h}(k)$ of the function $h(x)$ is defined by

$$\widetilde{h}(k)=\int_{-\infty}^{\infty}h(x)e^{-ikx}dx.$$

(i) State the inverse Fourier transform formula expressing $h(x)$ in terms of $\widetilde{h}(k)$.

(ii) State the convolution theorem for Fourier transforms.

(iii) Find the Fourier transform of the function $f(x)=e^{-|x|}$. Hence show that the convolution of the function $f(x)=e^{-|x|}$ with itself is given by the integral expression

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{(1+k^2)^2}dk.$$

Paper 4, Section I

5A Methods

Use the method of characteristics to find a continuous solution $u(x,y)$ of the equation

$$y\frac{\partial u}{\partial x}+x\frac{\partial u}{\partial y}=0,$$

subject to the condition $u(0,y)=y^4$.

In which region of the plane is the solution uniquely determined?

Paper 1, Section II

14A Methods

Let $f(t)$ be a real function defined on an interval $(-T,T)$ with Fourier series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{T}+b_n\sin\frac{n\pi t}{T}\right).$$

State and prove Parseval's theorem for $f(t)$ and its Fourier series. Write down the formulae for a_0, a_n and b_n in terms of $f(t)$, $\cos\frac{n\pi t}{T}$ and $\sin\frac{n\pi t}{T}$.

Find the Fourier series of the square wave function defined on $(-\pi,\pi)$ by

$$g(t)=\begin{cases}0 & -\pi<t\leqslant 0\\ 1 & \quad 0<t<\pi.\end{cases}$$

Hence evaluate

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}.$$

Using some of the above results evaluate

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}.$$

What is the sum of the Fourier series for $g(t)$ at $t=0$? Comment on your answer.

Paper 2, Section II

16A Methods

Use a Green's function to find an integral expression for the solution of the equation

$$\frac{d^2\theta}{dt^2}+4\frac{d\theta}{dt}+29\,\theta=f(t)$$

for $t\geqslant 0$ subject to the initial conditions

$$\theta(0)=0\quad\text{and}\quad\frac{d\theta}{dt}(0)=0.$$

Paper 3, Section II

15A Methods

A uniform stretched string of length L, density per unit length μ and tension $T = \mu c^2$ is fixed at both ends. Its transverse displacement is given by $y(x,t)$ for $0 \leqslant x \leqslant L$. The motion of the string is resisted by the surrounding medium with a resistive force per unit length of $-2k\mu \dfrac{\partial y}{\partial t}$.

(i) Show that the equation of motion of the string is

$$\frac{\partial^2 y}{\partial t^2} + 2k \frac{\partial y}{\partial t} - c^2 \frac{\partial^2 y}{\partial x^2} = 0$$

provided that the transverse motion can be regarded as small.

(ii) Suppose now that $k = \dfrac{\pi c}{L}$. Find the displacement of the string for $t \geqslant 0$ given the initial conditions

$$y(x,0) = A \sin\left(\frac{\pi x}{L}\right) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0) = 0.$$

(iii) Sketch the transverse displacement at $x = \dfrac{L}{2}$ as a function of time for $t \geqslant 0$.

Paper 4, Section II

17A Methods

Let D be a two dimensional domain with boundary ∂D. Establish Green's second identity

$$\int_D (\phi\nabla^2\psi - \psi\nabla^2\phi)dA = \int_{\partial D}\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)ds$$

where $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative on ∂D.

State the differential equation and boundary conditions which are satisfied by a Dirichlet Green's function $G(\mathbf{r},\mathbf{r}_0)$ for the Laplace operator on the domain D, where $\mathbf{r}_0$ is a fixed point in the interior of D.

Suppose that $\nabla^2\psi = 0$ on D. Show that

$$\psi(\mathbf{r}_0) = \int_{\partial D}\psi(\mathbf{r})\frac{\partial}{\partial n}G(\mathbf{r},\mathbf{r_0})ds.$$

Consider Laplace's equation in the upper half plane,

$$\nabla^2\psi(x,y) = 0, \quad -\infty < x < \infty \quad \text{and} \quad y > 0,$$

with boundary conditions $\psi(x,0) = f(x)$ where $f(x) \to 0$ as $|x| \to \infty$, and $\psi(x,y) \to 0$ as $\sqrt{x^2+y^2} \to \infty$. Show that the solution is given by the integral formula

$$\psi(x_0,y_0) = \frac{y_0}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{(x-x_0)^2+y_0^2}\,dx.$$

[*Hint: It might be useful to consider*

$$G(\mathbf{r},\mathbf{r}_0) = \frac{1}{2\pi}(\log|\mathbf{r}-\mathbf{r}_0| - \log|\mathbf{r}-\tilde{\mathbf{r}}_0|)$$

for suitable $\tilde{\mathbf{r}}_\mathbf{0}$. *You may assume* $\nabla^2\log|\mathbf{r}-\mathbf{r}_0| = 2\pi\delta(\mathbf{r}-\mathbf{r}_0)$.]

Paper 2, Section I

4G Metric and Topological Spaces

(i) Let $t > 0$. For $\mathbf{x} = (x, y),\ \mathbf{x}' = (x', y') \in \mathbb{R}^2$, let

$$d(\mathbf{x}, \mathbf{x}') = |x' - x| + t|y' - y|,$$

$$\delta(\mathbf{x}, \mathbf{x}') = \sqrt{(x' - x)^2 + (y' - y)^2}.$$

(δ is the usual Euclidean metric on $\mathbb{R}^2$.) Show that d is a metric on $\mathbb{R}^2$ and that the two metrics d, δ give rise to the same topology on $\mathbb{R}^2$.

(ii) Give an example of a topology on $\mathbb{R}^2$, different from the one in (i), whose induced topology (subspace topology) on the x-axis is the usual topology (the one defined by the metric $d(x, x') = |x' - x|$). Justify your answer.

Paper 3, Section I

3G Metric and Topological Spaces

Let X, Y be topological spaces, and suppose Y is Hausdorff.

(i) Let $f, g : X \to Y$ be two continuous maps. Show that the set

$$E(f, g) := \{x \in X \mid f(x) = g(x)\} \subset X$$

is a closed subset of X.

(ii) Let W be a dense subset of X. Show that a continuous map $f : X \to Y$ is determined by its restriction $f|_W$ to W.

Paper 1, Section II

12G Metric and Topological Spaces

Let X be a metric space with the distance function $d : X \times X \to \mathbb{R}$. For a subset Y of X, its *diameter* is defined as $\delta(Y) := \sup\{d(y, y') \mid y, y' \in Y\}$.

Show that, if X is compact and $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open covering of X, then there exists an $\epsilon > 0$ such that every subset $Y \subset X$ with $\delta(Y) < \epsilon$ is contained in some U_λ.

Paper 4, Section II

13G Metric and Topological Spaces

Let X, Y be topological spaces and $X \times Y$ their product set. Let $p_Y : X \times Y \to Y$ be the projection map.

(i) Define the product topology on $X \times Y$. Prove that if a subset $Z \subset X \times Y$ is open then $p_Y(Z)$ is open in Y.

(ii) Give an example of X, Y and a closed set $Z \subset X \times Y$ such that $p_Y(Z)$ is not closed.

(iii) When X is compact, show that if a subset $Z \subset X \times Y$ is closed then $p_Y(Z)$ is closed.

Paper 1, Section I

6B Numerical Analysis

Orthogonal monic polynomials $p_0, p_1, \ldots, p_n, \ldots$ are defined with respect to the inner product $\langle p, q\rangle = \int_{-1}^{1} w(x)p(x)q(x)\,dx$, where p_n is of degree n. Show that such polynomials obey a three-term recurrence relation

$$p_{n+1}(x) = (x-\alpha_n)p_n(x) - \beta_n p_{n-1}(x)$$

for appropriate choices of α_n and β_n.

Now suppose that $w(x)$ is an even function of x. Show that the p_n are even or odd functions of x according to whether n is even or odd.

Paper 4, Section I

8B Numerical Analysis

Consider the multistep method for numerical solution of the differential equation $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$:

$$\sum_{l=0}^{s} \rho_l \mathbf{y}_{n+l} = h \sum_{l=0}^{s} \sigma_l \mathbf{f}(t_{n+l}, \mathbf{y}_{n+l}), \quad n = 0, 1, \ldots .$$

What does it mean to say that the method is of order p, and that the method is convergent?

Show that the method is of order p if

$$\sum_{l=0}^{s} \rho_l = 0, \quad \sum_{l=0}^{s} l^k \rho_l = k \sum_{l=0}^{s} l^{k-1}\sigma_l, \quad k = 1, 2, \ldots, p,$$

and give the conditions on $\rho(w) = \sum_{l=0}^{s} \rho_l w^l$ that ensure convergence.

Hence determine for what values of θ and the σ_i the two-step method

$$\mathbf{y}_{n+2} - (1-\theta)\mathbf{y}_{n+1} - \theta\mathbf{y}_n = h[\sigma_0 \mathbf{f}(t_n, \mathbf{y}_n) + \sigma_1 \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) + \sigma_2 \mathbf{f}(t_{n+2}, \mathbf{y}_{n+2})]$$

is (a) convergent, and (b) of order 3.

Paper 1, Section II
18B Numerical Analysis

Consider a function $f(x)$ defined on the domain $x \in [0,1]$. Find constants α, β, γ so that for any fixed $\xi \in [0,1]$,

$$f''(\xi) = \alpha f(0) + \beta f'(0) + \gamma f(1)$$

is exactly satisfied for polynomials of degree less than or equal to two.

By using the Peano kernel theorem, or otherwise, show that

$$\begin{aligned} f'(\xi) - f'(0) - \xi\left(\alpha f(0) + \beta f'(0) + \gamma f(1)\right) &= \int_0^\xi (\xi - \theta) H_1(\theta) f'''(\theta)\, d\theta \\ &\quad + \int_0^\xi \theta H_2(\theta) f'''(\theta)\, d\theta + \int_\xi^1 \xi H_2(\theta) f'''(\theta)\, d\theta, \end{aligned}$$

where $H_1(\theta) = 1 - (1-\theta)^2 \geqslant 0,\ H_2(\theta) = -(1-\theta)^2 \leqslant 0$. Thus show that

$$\left| f'(\xi) - f'(0) - \xi(\alpha f(0) + \beta f'(0) + \gamma f(1)) \right| \leqslant \frac{1}{6}(2\xi - 3\xi^2 + 4\xi^3 - \xi^4)\, \|f'''\|_\infty .$$

Paper 2, Section II
19B Numerical Analysis

What is the QR-decomposition of a matrix A? Explain how to construct the matrices Q and R by the Gram-Schmidt procedure, and show how the decomposition can be used to solve the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ when A is a square matrix.

Why is this procedure not useful for numerical decomposition of large matrices? Give a brief description of an alternative procedure using Givens rotations.

Find a QR-decomposition for the matrix

$$\mathsf{A} = \begin{bmatrix} 3 & 4 & 7 & 13 \\ -6 & -8 & -8 & -12 \\ 3 & 4 & 7 & 11 \\ 0 & 2 & 5 & 7 \end{bmatrix}.$$

Is your decomposition unique? Use the decomposition you have found to solve the equation

$$\mathsf{A}\mathbf{x} = \begin{bmatrix} 4 \\ 6 \\ 2 \\ 9 \end{bmatrix}.$$

Paper 3, Section II

19B Numerical Analysis

A Gaussian quadrature formula provides an approximation to the integral

$$\int_{-1}^{1} (1-x^2) f(x)\, dx \approx \sum_{k=1}^{\nu} b_k f(c_k)$$

which is exact for all $f(x)$ that are polynomials of degree $\leqslant (2\nu - 1)$.

Write down explicit expressions for the b_k in terms of integrals, and explain why it is necessary that the c_k are the zeroes of a (monic) polynomial p_ν of degree ν that satisfies $\int_{-1}^{1}(1-x^2)p_\nu(x)q(x)\, dx = 0$ for any polynomial $q(x)$ of degree less than ν.

The first such polynomials are $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - 1/5$, $p_3 = x^3 - 3x/7$. Show that the Gaussian quadrature formulae for $\nu = 2, 3$ are

$$\begin{aligned} \nu = 2: &\quad \frac{2}{3}\left[f(-\tfrac{1}{\sqrt{5}}) + f(\tfrac{1}{\sqrt{5}}) \right], \\ \nu = 3: &\quad \frac{14}{45}\left[f(-\sqrt{\tfrac{3}{7}}) + f(\sqrt{\tfrac{3}{7}}) \right] + \frac{32}{45} f(0). \end{aligned}$$

Verify the result for $\nu = 3$ by considering $f(x) = 1, x^2, x^4$.

Paper 1, Section I

8H Optimization

Suppose that $Ax \leqslant b$ and $x \geqslant 0$ and $A^T y \geqslant c$ and $y \geqslant 0$ where x and c are n-dimensional column vectors, y and b are m-dimensional column vectors, and A is an $m \times n$ matrix. Here, the vector inequalities are interpreted component-wise.

(i) Show that $c^T x \leqslant b^T y$.

(ii) Find the maximum value of

$$6x_1 + 8x_2 + 3x_3 \quad \text{subject to} \quad \begin{aligned} &2x_1 + 4x_2 + x_3 \leqslant 10, \\ &3x_1 + 4x_2 + 3x_3 \leqslant 6, \\ &x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

You should state any results from the course used in your solution.

Paper 2, Section I

9H Optimization

Let $N = \{1, \ldots, n\}$ be the set of nodes of a network, where 1 is the source and n is the sink. Let c_{ij} denote the capacity of the arc from node i to node j.

(i) In the context of maximising the flow through this network, define the following terms: feasible flow, flow value, cut, cut capacity.

(ii) State and prove the max-flow min-cut theorem for network flows.

Paper 3, Section II

21H Optimization

(i) What does it mean to say a set $C \subseteq \mathbb{R}^n$ is convex?

(ii) What does it mean to say z is an extreme point of a convex set C?

Let A be an $m \times n$ matrix, where $n > m$. Let b be an $m \times 1$ vector, and let

$$C = \{x \in \mathbb{R}^n : Ax = b, x \geqslant 0\}$$

where the inequality is interpreted component-wise.

(iii) Show that C is convex.

(iv) Let $z = (z_1, \ldots, z_n)^T$ be a point in C with the property that at least $m+1$ indices i are such that $z_i > 0$. Show that z is not an extreme point of C. [*Hint: If $r > m$, then any set of r vectors in $\mathbb{R}^m$ is linearly dependent.*]

(v) Now suppose that every set of m columns of A is linearly independent. Let $z = (z_1, \ldots, z_n)^T$ be a point in C with the property that at most m indices i are such that $z_i > 0$. Show that z is an extreme point of C.

Paper 4, Section II

20H Optimization

A company must ship coal from four mines, labelled A, B, C, D, to supply three factories, labelled a, b, c. The per unit transport cost, the outputs of the mines, and the requirements of the factories are given below.

	A	B	C	D	
a	12	3	5	2	34
b	4	11	2	6	21
c	3	9	7	4	23
	20	32	15	11	

For instance, mine B can produce 32 units of coal, factory a requires 34 units of coal, and it costs 3 units of money to ship one unit of coal from B to a. What is the minimal cost of transporting coal from the mines to the factories?

Now suppose increased efficiency allows factory b to reduce its requirement to 20.8 units of coal, and as a consequence, mine B reduces its output to 31.8 units. By how much does the transport cost decrease?

Paper 3, Section I

8C Quantum Mechanics

A particle of mass m and energy E, incident from $x = -\infty$, scatters off a delta function potential at $x = 0$. The time independent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\delta(x)\psi = E\psi$$

where U is a positive constant. Find the reflection and transmission probabilities.

Paper 4, Section I

6C Quantum Mechanics

Consider the 3-dimensional oscillator with Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\omega^2}{2}(x^2 + y^2 + 4z^2)\,.$$

Find the ground state energy and the spacing between energy levels. Find the degeneracies of the lowest three energy levels.

[You may assume that the energy levels of the 1-dimensional harmonic oscillator with Hamiltonian

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2$$

are $(n + \frac{1}{2})\hbar\omega$, $n = 0, 1, 2, \ldots$.]

Paper 1, Section II

15C Quantum Mechanics

For a quantum mechanical particle moving freely on a circle of length 2π, the wavefunction $\psi(t,x)$ satisfies the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

on the interval $0 \leqslant x \leqslant 2\pi$, and also the periodicity conditions $\psi(t,2\pi) = \psi(t,0)$, and $\dfrac{\partial\psi}{\partial x}(t,2\pi) = \dfrac{\partial\psi}{\partial x}(t,0)$. Find the allowed energy levels of the particle, and their degeneracies.

The current is defined as

$$j = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

where ψ is a normalized state. Write down the general normalized state of the particle when it has energy $2\hbar^2/m$, and show that in any such state the current j is independent of x and t. Find a state with this energy for which the current has its maximum positive value, and find a state with this energy for which the current vanishes.

Paper 2, Section II

17C Quantum Mechanics

The quantum mechanical angular momentum operators are

$$L_i = -i\hbar\,\epsilon_{ijk}\,x_j\,\frac{\partial}{\partial x_k} \qquad (i = 1,2,3).$$

Show that each of these is hermitian.

The total angular momentum operator is defined as $\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2$. Show that $\langle\mathbf{L}^2\rangle \geqslant \langle L_3^2\rangle$ in any state, and show that the only states where $\langle\mathbf{L}^2\rangle = \langle L_3^2\rangle$ are those with no angular dependence. Verify that the eigenvalues of the operators $\mathbf{L}^2$ and L_3^2 (whose values you may quote without proof) are consistent with these results.

Paper 3, Section II

16C Quantum Mechanics

For an electron in a hydrogen atom, the stationary state wavefunctions are of the form $\psi(r,\theta,\phi) = R(r)Y_{lm}(\theta,\phi)$, where in suitable units R obeys the radial equation

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R + 2\left(E + \frac{1}{r}\right)R = 0\,.$$

Explain briefly how the terms in this equation arise.

This radial equation has bound state solutions of energy $E = E_n$, where $E_n = -\frac{1}{2n^2}$ $(n = 1,2,3,\dots)$. Show that when $l = n-1$, there is a solution of the form $R(r) = r^{\alpha}e^{-r/n}$, and determine α. Find the expectation value $\langle r \rangle$ in this state.

What is the total degeneracy of the energy level with energy E_n?

Paper 1, Section I
7H Statistics

Consider the experiment of tossing a coin n times. Assume that the tosses are independent and the coin is biased, with unknown probability p of heads and $1-p$ of tails. A total of X heads is observed.

(i) What is the maximum likelihood estimator $\widehat{p}$ of p?

Now suppose that a Bayesian statistician has the $\text{Beta}(M,N)$ prior distribution for p.

(ii) What is the posterior distribution for p?

(iii) Assuming the loss function is $L(p,a) = (p-a)^2$, show that the statistician's point estimate for p is given by

$$\frac{M+X}{M+N+n}.$$

[The $\text{Beta}(M,N)$ distribution has density $\dfrac{\Gamma(M+N)}{\Gamma(M)\Gamma(N)}x^{M-1}(1-x)^{N-1}$ for $0<x<1$ and mean $\dfrac{M}{M+N}$.]

Paper 2, Section I
8H Statistics

Let $X_1,\ldots,X_n$ be random variables with joint density function $f(x_1,\ldots,x_n;\theta)$, where θ is an unknown parameter. The null hypothesis $H_0:\theta=\theta_0$ is to be tested against the alternative hypothesis $H_1:\theta=\theta_1$.

(i) Define the following terms: critical region, Type I error, Type II error, size, power.

(ii) State and prove the Neyman–Pearson lemma.

Paper 1, Section II

19H Statistics

Let $X_1, \ldots, X_n$ be independent random variables with probability mass function $f(x;\theta)$, where θ is an unknown parameter.

(i) What does it mean to say that T is a sufficient statistic for θ? State, but do not prove, the factorisation criterion for sufficiency.

(ii) State and prove the Rao–Blackwell theorem.

Now consider the case where $f(x;\theta) = \dfrac{1}{x!}(-\log\theta)^x\theta$ for non-negative integer x and $0 < \theta < 1$.

(iii) Find a one-dimensional sufficient statistic T for θ.

(iv) Show that $\widetilde{\theta} = \mathbb{1}_{\{X_1=0\}}$ is an unbiased estimator of θ.

(v) Find another unbiased estimator $\widehat{\theta}$ which is a function of the sufficient statistic T and that has smaller variance than $\widetilde{\theta}$. You may use the following fact without proof: $X_1 + \cdots + X_n$ has the Poisson distribution with parameter $-n\log\theta$.

Paper 3, Section II
20H Statistics

Consider the general linear model

$$Y = X\beta + \epsilon$$

where X is a known $n \times p$ matrix, β is an unknown $p \times 1$ vector of parameters, and ϵ is an $n \times 1$ vector of independent $N(0, \sigma^2)$ random variables with unknown variance σ^2. Assume the $p \times p$ matrix $X^T X$ is invertible.

(i) Derive the least squares estimator $\widehat{\beta}$ of β.

(ii) Derive the distribution of $\widehat{\beta}$. Is $\widehat{\beta}$ an unbiased estimator of β?

(iii) Show that $\frac{1}{\sigma^2}\|Y - X\widehat{\beta}\|^2$ has the χ^2 distribution with k degrees of freedom, where k is to be determined.

(iv) Let $\widetilde{\beta}$ be an unbiased estimator of β of the form $\widetilde{\beta} = CY$ for some $p \times n$ matrix C. By considering the matrix $\mathbb{E}[(\widehat{\beta} - \widetilde{\beta})(\widehat{\beta} - \beta)^T]$ or otherwise, show that $\widehat{\beta}$ and $\widehat{\beta} - \widetilde{\beta}$ are independent.

[You may use standard facts about the multivariate normal distribution as well as results from linear algebra, including the fact that $I - X(X^T X)^{-1} X^T$ is a projection matrix of rank $n - p$, as long as they are carefully stated.]

Paper 4, Section II
19H Statistics

Consider independent random variables $X_1, \ldots, X_n$ with the $N(\mu_X, \sigma_X^2)$ distribution and $Y_1, \ldots, Y_n$ with the $N(\mu_Y, \sigma_Y^2)$ distribution, where the means μ_X, μ_Y and variances σ_X^2, σ_Y^2 are unknown. Derive the generalised likelihood ratio test of size α of the null hypothesis $H_0 : \sigma_X^2 = \sigma_Y^2$ against the alternative $H_1 : \sigma_X^2 \neq \sigma_Y^2$. Express the critical region in terms of the statistic $T = \dfrac{S_{XX}}{S_{XX} + S_{YY}}$ and the quantiles of a beta distribution, where

$$S_{XX} = \sum_{i=1}^{n} X_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^2 \text{ and } S_{YY} = \sum_{i=1}^{n} Y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} Y_i\right)^2 .$$

[You may use the following fact: if $U \sim \Gamma(a, \lambda)$ and $V \sim \Gamma(b, \lambda)$ are independent, then $\dfrac{U}{U+V} \sim \mathrm{Beta}(a, b)$.]

Paper 1, Section I

4D Variational Principles

(i) Write down the Euler-Lagrange equations for the volume integral

$$\int_V \big(\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u + 12u\big)dV\,,$$

where V is the unit ball $x^2+y^2+z^2 \leqslant 1$, and verify that the function $u(x,y,z) = x^2+y^2+z^2$ gives a stationary value of the integral subject to the condition $u = 1$ on the boundary.

(ii) Write down the Euler-Lagrange equations for the integral

$$\int_0^1 \big(\dot{x}^2 + \dot{y}^2 + 4x + 4y\big)\,dt\,,$$

where the dot denotes differentiation with respect to t, and verify that the functions $x(t) = t^2$, $y(t) = t^2$ give a stationary value of the integral subject to the boundary conditions $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$.

Paper 3, Section I

6D Variational Principles

Find, using a Lagrange multiplier, the four stationary points in $\mathbb{R}^3$ of the function $x^2 + y^2 + z^2$ subject to the constraint $x^2 + 2y^2 - z^2 = 1$. By considering the situation geometrically, or otherwise, identify the nature of the constrained stationary points.

How would your answers differ if, instead, the stationary points of the function $x^2 + 2y^2 - z^2$ were calculated subject to the constraint $x^2 + y^2 + z^2 = 1$?

Paper 2, Section II

15D Variational Principles

(i) Let $I[y] = \int_0^1 \big((y')^2 - y^2\big)dx$, where y is twice differentiable and $y(0) = y(1) = 0$. Write down the associated Euler-Lagrange equation and show that the only solution is $y(x) = 0$.

(ii) Let $J[y] = \int_0^1 (y' + y \tan x)^2 dx$, where y is twice differentiable and $y(0) = y(1) = 0$. Show that $J[y] = 0$ only if $y(x) = 0$.

(iii) Show that $I[y] = J[y]$ and deduce that the extremal value of $I[y]$ is a global minimum.

(iv) Use the second variation of $I[y]$ to verify that the extremal value of $I[y]$ is a local minimum.

(v) How would your answers to part (i) differ in the case $I[y] = \int_0^{2\pi} \big((y')^2 - y^2\big)dx$, where $y(0) = y(2\pi) = 0$? Show that the solution $y(x) = 0$ is not a global minimizer in this case. (You may use without proof the result $I[x(2\pi - x)] = -\frac{8}{15}(2\pi^2 - 5)$.) Explain why the arguments of parts (iii) and (iv) cannot be used.

Paper 4, Section II

16D Variational Principles

Derive the Euler-Lagrange equation for the integral

$$\int_{x_0}^{x_1} f(y, y', y'', x)\, dx$$

where the endpoints are fixed, and $y(x)$ and $y'(x)$ take given values at the endpoints.

Show that the only function $y(x)$ with $y(0) = 1$, $y'(0) = 2$ and $y(x) \to 0$ as $x \to \infty$ for which the integral

$$\int_0^\infty \big(y^2 + (y')^2 + (y' + y'')^2\big)dx$$

is stationary is $(3x + 1)e^{-x}$.