UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2010

List of Courses

Paper 2, Section I

3G Analysis II

Let $c > 1$ be a real number, and let F_c be the space of sequences $\mathbf{a} = (a_1, a_2, \dots)$ of real numbers a_i with $\sum_{r=1}^{\infty} c^{-r}|a_r|$ convergent. Show that $\|\mathbf{a}\|_c = \sum_{r=1}^{\infty} c^{-r}|a_r|$ defines a norm on F_c.

Let F denote the space of sequences $\mathbf{a}$ with $|a_i|$ bounded; show that $F \subset F_c$. If $c' > c$, show that the norms on F given by restricting to F the norms $\|\,.\,\|_c$ on F_c and $\|\,.\,\|_{c'}$ on $F_{c'}$ are not Lipschitz equivalent.

By considering sequences of the form $\mathbf{a}^{(n)} = (a,\, a^2,\, \dots,\, a^n,\, 0,\, 0,\, \dots)$ in F, for a an appropriate real number, or otherwise, show that F (equipped with the norm $\|\,.\,\|_c$) is not complete.

Paper 3, Section I

2G Analysis II

Consider the map $f : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$f(x,\, y,\, z) \;=\; (x + y + z,\; xy + yz + zx,\; xyz)\,.$$

Show that f is differentiable everywhere and find its derivative.

Stating carefully any theorem that you quote, show that f is locally invertible near a point (x, y, z) unless $(x - y)(y - z)(z - x) = 0$.

Paper 4, Section I

3G Analysis II

Let S denote the set of continuous real-valued functions on the interval $[0, 1]$. For $f, g \in S$, set

$$d_1(f, g) = \sup\{|f(x) - g(x)| \,:\, x \in [0, 1]\} \quad \text{and} \quad d_2(f, g) = \int_0^1 |f(x) - g(x)|\, dx\,.$$

Show that both d_1 and d_2 define metrics on S. Does the identity map on S define a continuous map of metric spaces $(S, d_1) \to (S, d_2)$? Does the identity map define a continuous map of metric spaces $(S, d_2) \to (S, d_1)$?

Paper 1, Section II

11G Analysis II

State and prove the contraction mapping theorem. Demonstrate its use by showing that the differential equation $f'(x) = f(x^2)$, with boundary condition $f(0) = 1$, has a unique solution on $[0, 1)$, with one-sided derivative $f'(0) = 1$ at zero.

Paper 2, Section II

12G Analysis II

Suppose the functions f_n $(n = 1, 2, \ldots)$ are defined on the open interval $(0, 1)$ and that f_n tends uniformly on $(0, 1)$ to a function f. If the f_n are continuous, show that f is continuous. If the f_n are differentiable, show by example that f need not be differentiable.

Assume now that each f_n is differentiable and the derivatives f_n' converge uniformly on $(0, 1)$. For any given $c \in (0, 1)$, we define functions $g_{c,n}$ by

$$g_{c,n}(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x - c} & \text{for } x \neq c, \\ f_n'(c) & \text{for } x = c. \end{cases}$$

Show that each $g_{c,n}$ is continuous. Using the general principle of uniform convergence (the Cauchy criterion) and the Mean Value Theorem, or otherwise, prove that the functions $g_{c,n}$ converge uniformly to a continuous function g_c on $(0, 1)$, where

$$g_c(x) = \frac{f(x) - f(c)}{x - c} \quad \text{for } x \neq c \ .$$

Deduce that f is differentiable on $(0, 1)$.

Paper 3, Section II

12G Analysis II

Let $f : U \to \mathbf{R}^n$ be a map on an open subset $U \subset \mathbf{R}^m$. Explain what it means for f to be differentiable on U. If $g : V \to \mathbf{R}^m$ is a differentiable map on an open subset $V \subset \mathbf{R}^p$ with $g(V) \subset U$, state and prove the Chain Rule for the derivative of the composite fg.

Suppose now $F : \mathbf{R}^n \to \mathbf{R}$ is a differentiable function for which the partial derivatives $D_1F(\mathbf{x}) = D_2F(\mathbf{x}) = \ldots = D_nF(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^n$. By considering the function $G : \mathbf{R}^n \to \mathbf{R}$ given by

$$G(y_1, \ldots, y_n) = F\left(y_1, \ldots, y_{n-1}, y_n - \sum_{i=1}^{n-1} y_i\right),$$

or otherwise, show that there exists a differentiable function $h : \mathbf{R} \to \mathbf{R}$ with $F(x_1, \ldots, x_n) = h(x_1 + \cdots + x_n)$ at all points of $\mathbf{R}^n$.

Paper 4, Section II

12G Analysis II

What does it mean to say that a function f on an interval in $\mathbf{R}$ is *uniformly continuous*? Assuming the Bolzano–Weierstrass theorem, show that any continuous function on a finite closed interval is uniformly continuous.

Suppose that f is a continuous function on the real line, and that $f(x)$ tends to finite limits as $x \to \pm\infty$; show that f is uniformly continuous.

If f is a uniformly continuous function on $\mathbf{R}$, show that $f(x)/x$ is bounded as $x \to \pm\infty$. If g is a continuous function on $\mathbf{R}$ for which $g(x)/x \to 0$ as $x \to \pm\infty$, determine whether g is necessarily uniformly continuous, giving proof or counterexample as appropriate.

Paper 4, Section I

4G Complex Analysis

State the principle of the argument for meromorphic functions and show how it follows from the Residue theorem.

Paper 3, Section II

13G Complex Analysis

State Morera's theorem. Suppose f_n $(n = 1, 2, \ldots)$ are analytic functions on a domain $U \subset \mathbf{C}$ and that f_n tends locally uniformly to f on U. Show that f is analytic on U. Explain briefly why the derivatives f_n' tend locally uniformly to f'.

Suppose now that the f_n are nowhere vanishing and f is not identically zero. Let a be any point of U; show that there exists a closed disc $\overline{\Delta} \subset U$ with centre a, on which the convergence of f_n and f_n' are both uniform, and where f is nowhere zero on $\overline{\Delta} \setminus \{a\}$. By considering

$$\frac{1}{2\pi i} \int_C \frac{f_n'(w)}{f_n(w)}\, dw$$

(where C denotes the boundary of $\overline{\Delta}$), or otherwise, deduce that $f(a) \neq 0$.

Paper 1, Section I

2A Complex Analysis or Complex Methods

(a) Write down the definition of the complex derivative of the function $f(z)$ of a single complex variable.

(b) Derive the Cauchy-Riemann equations for the real and imaginary parts $u(x,y)$ and $v(x,y)$ of $f(z)$, where $z = x + iy$ and

$$f(z) = u(x,y) + iv(x,y).$$

(c) State necessary and sufficient conditions on $u(x,y)$ and $v(x,y)$ for the function $f(z)$ to be complex differentiable.

Paper 1, Section II

13A Complex Analysis or Complex Methods

Calculate the following real integrals by using contour integration. Justify your steps carefully.

(a)

$$I_1 = \int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx, \qquad a > 0,$$

(b)

$$I_2 = \int_0^\infty \frac{x^{1/2} \log x}{1 + x^2}\, dx.$$

Paper 2, Section II

13A Complex Analysis or Complex Methods

(a) Prove that a complex differentiable map, $f(z)$, is conformal, i.e. preserves angles, provided a certain condition holds on the first complex derivative of $f(z)$.

(b) Let D be the region

$$D := \{z \in \mathbb{C} : |z-1| > 1 \text{ and } |z-2| < 2\}.$$

Draw the region D. It might help to consider the two sets

$$C(1) := \{z \in \mathbb{C} : |z-1| = 1\},$$
$$C(2) := \{z \in \mathbb{C} : |z-2| = 2\}.$$

(c) For the transformations below identify the images of D.

Step 1: The first map is $f_1(z) = \dfrac{z-1}{z}$,

Step 2: The second map is the composite $f_2 f_1$ where $f_2(z) = (z - \frac{1}{2})i$,

Step 3: The third map is the composite $f_3 f_2 f_1$ where $f_3(z) = e^{2\pi z}$.

(d) Write down the inverse map to the composite $f_3 f_2 f_1$, explaining any choices of branch.

[*The composite* $f_2 f_1$ *means* $f_2(f_1(z))$.]

Paper 3, Section I

4A Complex Methods

(a) Prove that the real and imaginary parts of a complex differentiable function are harmonic.

(b) Find the most general harmonic polynomial of the form

$$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

where a, b, c, d, x and y are real.

(c) Write down a complex analytic function of $z = x + iy$ of which $u(x,y)$ is the real part.

Paper 4, Section II

14A Complex Methods

A linear system is described by the differential equation

$$y'''(t) - y''(t) - 2\,y'(t) + 2\,y(t) \;=\; f(t)\,,$$

with initial conditions

$$y\,(0) \;=\; 0\,, \qquad y'(0) \;=\; 1\,, \qquad y''(0) \;=\; 1\,.$$

The Laplace transform of $f(t)$ is defined as

$$\mathcal{L}[f(t)] = \tilde{f}(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

You may assume the following Laplace transforms,

$$\begin{aligned}
\mathcal{L}[y(t)] &= \tilde{y}(s),\\
\mathcal{L}[y'(t)] &= s\tilde{y}(s) - y(0),\\
\mathcal{L}[y''(t)] &= s^2\tilde{y}(s) - sy(0) - y'(0),\\
\mathcal{L}[y'''(t)] &= s^3\tilde{y}(s) - s^2y(0) - sy'(0) - y''(0).
\end{aligned}$$

(a) Use Laplace transforms to determine the response, $y_1(t)$, of the system to the signal

$$f(t) = -2.$$

(b) Determine the response, $y_2(t)$, given that its Laplace transform is

$$\tilde{y}_2(s) = \frac{1}{s^2(s-1)^2}.$$

(c) Given that

$$y'''(t) - y''(t) - 2y'(t) + 2y(t) \;=\; g(t)$$

leads to the response with Laplace transform

$$\tilde{y}(s) \;=\; \frac{1}{s^2(s-1)^2}\,,$$

determine $g(t)$.

Paper 2, Section I

6C Electromagnetism

Write down Maxwell's equations for electromagnetic fields in a non-polarisable and non-magnetisable medium.

Show that the homogenous equations (those not involving charge or current densities) can be solved in terms of vector and scalar potentials $\mathbf{A}$ and ϕ.

Then re-express the inhomogeneous equations in terms of $\mathbf{A}$, ϕ and $f = \boldsymbol{\nabla}\cdot\mathbf{A} + c^{-2}\dot{\phi}$. Show that the potentials can be chosen so as to set $f = 0$ and hence rewrite the inhomogeneous equations as wave equations for the potentials. [*You may assume that the inhomogeneous wave equation* $\nabla^2\psi - c^{-2}\ddot{\psi} = \sigma(\mathbf{x}, t)$ *always has a solution* $\psi(\mathbf{x}, t)$ *for any given* $\sigma(\mathbf{x}, t)$.]

Paper 4, Section I

7B Electromagnetism

Give an expression for the force $\mathbf{F}$ on a charge q moving at velocity $\mathbf{v}$ in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. Consider a stationary electric circuit C, and let S be a stationary surface bounded by C. Derive from Maxwell's equations the result

$$\mathcal{E} = -\frac{d\Phi}{dt} \qquad (*)$$

where the electromotive force $\mathcal{E} = \oint_C q^{-1}\mathbf{F}\cdot d\mathbf{r}$ and the flux $\Phi = \int_S \mathbf{B}\cdot d\mathbf{S}$.

Now assume that $(*)$ also holds for a moving circuit. A circular loop of wire of radius a and total resistance R, whose normal is in the z-direction, moves at constant speed v in the x-direction in the presence of a magnetic field $\mathbf{B} = (0, 0, B_0\, x/d)$. Find the current in the wire.

Paper 1, Section II

16C Electromagnetism

A capacitor consists of three perfectly conducting coaxial cylinders of radii a, b and c where $0 < a < b < c$, and length L where $L \gg c$ so that end effects may be ignored. The inner and outer cylinders are maintained at zero potential, while the middle cylinder is held at potential V. Assuming its cylindrical symmetry, compute the electrostatic potential within the capacitor, the charge per unit length on the middle cylinder, the capacitance and the electrostatic energy, both per unit length.

Next assume that the radii a and c are fixed, as is the potential V, while the radius b is allowed to vary. Show that the energy achieves a minimum when b is the geometric mean of a and c.

Paper 2, Section II

18C Electromagnetism

A steady current I_2 flows around a loop $\mathcal{C}_2$ of a perfectly conducting narrow wire. Assuming that the gauge condition $\nabla \cdot \mathbf{A} = 0$ holds, the vector potential at points away from the loop may be taken to be

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I_2}{4\pi} \oint_{\mathcal{C}_2} \frac{d\mathbf{r}_2}{|\mathbf{r} - \mathbf{r}_2|} .$$

First verify that the gauge condition is satisfied here. Then obtain the Biot-Savart formula for the magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I_2}{4\pi} \oint_{\mathcal{C}_2} \frac{d\mathbf{r}_2 \times (\mathbf{r} - \mathbf{r}_2)}{|\mathbf{r} - \mathbf{r}_2|^3} .$$

Next suppose there is a similar but separate loop $\mathcal{C}_1$ with current I_1. Show that the magnetic force exerted on loop $\mathcal{C}_1$ by loop $\mathcal{C}_2$ is

$$\mathbf{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} d\mathbf{r}_1 \times \left(d\mathbf{r}_2 \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \right) .$$

Is this consistent with Newton's third law? Justify your answer.

Paper 3, Section II

17C Electromagnetism

Write down Maxwell's equations in a region with no charges and no currents. Show that if $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ is a solution then so is $\widetilde{\mathbf{E}}(\mathbf{x},t) = c\mathbf{B}(\mathbf{x},t)$ and $\widetilde{\mathbf{B}}(\mathbf{x},t) = -\mathbf{E}(\mathbf{x},t)/c$. Write down the boundary conditions on $\mathbf{E}$ and $\mathbf{B}$ at the boundary with unit normal $\mathbf{n}$ between a perfect conductor and a vacuum.

Electromagnetic waves propagate inside a tube of perfectly conducting material. The tube's axis is in the z-direction, and it is surrounded by a vacuum. The fields may be taken to be the real parts of

$$\mathbf{E}(\mathbf{x},t) \,=\, \mathbf{e}(x,y)e^{\,i(kz-\omega t)}, \qquad \mathbf{B}(\mathbf{x},t) \,=\, \mathbf{b}(x,y)e^{\,i(kz-\omega t)} \,.$$

Write down Maxwell's equations in terms of $\mathbf{e}$, $\mathbf{b}$, k and ω.

Suppose first that $b_z(x,y) = 0$. Show that the solution is determined by

$$\mathbf{e} \,=\, \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}\,,\, ik\left[1 - \frac{\omega^2}{k^2c^2}\right]\psi\right),$$

where the function $\psi(x,y)$ satisfies

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \gamma^2\psi \,=\, 0\,,$$

and ψ vanishes on the boundary of the tube. Here γ^2 is a constant whose value should be determined.

Obtain a similar condition for the case where $e_z(x,y) = 0$. [*You may find it useful to use a result from the first paragraph.*] What is the corresponding boundary condition?

Paper 1, Section I

5B Fluid Dynamics

A planar solenoidal velocity field has the velocity potential

$$\phi(x, y, t) = xe^{-t} + ye^{t}.$$

Find and sketch (i) the streamlines at $t = 0$; (ii) the pathline that passes through the origin at $t = 0$; (iii) the locus at $t = 0$ of points that pass through the origin at earlier times (streakline).

Paper 2, Section I

7B Fluid Dynamics

Write down an expression for the velocity field of a line vortex of strength κ.

Consider N identical line vortices of strength κ arranged at equal intervals round a circle of radius a. Show that the vortices all move around the circle at constant angular velocity $(N-1)\kappa/(4\pi a^2)$.

Paper 1, Section II

17B Fluid Dynamics

Starting with the Euler equations for an inviscid incompressible fluid, derive Bernoulli's theorem for unsteady irrotational flow.

Inviscid fluid of density ρ is contained within a U-shaped tube with the arms vertical, of height h and with the same (unit) cross-section. The ends of the tube are closed. In the equilibrium state the pressures in the two arms are p_1 and p_2 and the heights of the fluid columns are ℓ_1, ℓ_2.

The fluid in arm 1 is displaced upwards by a distance ξ (and in the other arm downward by the same amount). In the subsequent evolution the pressure above each column may be taken as inversely proportional to the length of tube above the fluid surface. Using Bernoulli's theorem, show that $\xi(t)$ obeys the equation

$$\rho(\ell_1 + \ell_2)\ddot{\xi} + \frac{p_1\xi}{h - \ell_1 - \xi} + \frac{p_2\xi}{h - \ell_2 + \xi} + 2\rho g\xi = 0.$$

Now consider the special case $\ell_1 = \ell_2 = \ell_0$, $p_1 = p_2 = p_0$. Construct a first integral of this equation and hence give an expression for the total kinetic energy $\rho\ell_0\dot{\xi}^2$ of the flow in terms of ξ and the maximum displacement ξ_{max}.

Paper 3, Section II
18B Fluid Dynamics

Write down the exact kinematic and dynamic boundary conditions that apply at the free surface $z = \eta(x, t)$ of a fluid layer in the presence of gravity in the z-direction. Show how these may be approximated for small disturbances of a hydrostatic state about $z = 0$. (The flow of the fluid is in the (x, z)-plane and may be taken to be irrotational, and the pressure at the free surface may be assumed to be constant.)

Fluid of density ρ fills the region $0 > z > -h$. At $z = -h$ the z-component of the velocity is $\epsilon \mathrm{Re}(e^{i\omega t} \cos kx)$, where $|\epsilon| \ll 1$. Find the resulting disturbance of the free surface, assuming this to be small. Explain physically why your answer has a singularity for a particular value of ω^2.

Paper 4, Section II
18B Fluid Dynamics

Write down the velocity potential for a line source flow of strength m located at $(r, \theta) = (d, 0)$ in polar coordinates (r, θ) and derive the velocity components u_r, u_θ.

A two-dimensional flow field consists of such a source in the presence of a circular cylinder of radius a $(a < d)$ centred at the origin. Show that the flow field outside the cylinder is the sum of the original source flow, together with that due to a source of the same strength at $(a^2/d, 0)$ and another at the origin, of a strength to be determined.

Use Bernoulli's law to find the pressure distribution on the surface of the cylinder, and show that the total force exerted on it is in the x-direction and of magnitude

$$\frac{m^2\rho}{2\pi^2} \int_0^{2\pi} \frac{ad^2 \sin^2\theta \cos\theta}{(a^2 + d^2 - 2\,ad\,\cos\theta)^2}\, d\theta\,,$$

where ρ is the density of the fluid. Without evaluating the integral, show that it is positive. Comment on the fact that the force on the cylinder is therefore *towards* the source.

Paper 1, Section I

3F Geometry

(i) Define the notion of curvature for surfaces embedded in $\mathbb{R}^3$.

(ii) Prove that the unit sphere in $\mathbb{R}^3$ has curvature $+1$ at all points.

Paper 3, Section I

5F Geometry

(i) Write down the Poincaré metric on the unit disc model D of the hyperbolic plane. Compute the hyperbolic distance ρ from $(0,0)$ to $(r,0)$, with $0<r<1$.

(ii) Given a point P in D and a hyperbolic line L in D with P not on L, describe how the minimum distance from P to L is calculated. Justify your answer.

Paper 2, Section II

14F Geometry

Suppose that $a>0$ and that $S\subset\mathbb{R}^3$ is the half-cone defined by $z^2=a(x^2+y^2)$, $z>0$. By using an explicit smooth parametrization of S, calculate the curvature of S.

Describe the geodesics on S. Show that for $a=3$, no geodesic intersects itself, while for $a>3$ some geodesic does so.

Paper 3, Section II

14F Geometry

Describe the hyperbolic metric on the upper half-plane H. Show that any Möbius transformation that preserves H is an isometry of this metric.

Suppose that $z_1,z_2\in H$ are distinct and that the hyperbolic line through z_1 and z_2 meets the real axis at w_1,w_2. Show that the hyperbolic distance $\rho(z_1,z_2)$ between z_1 and z_2 is given by $\rho(z_1,z_2)=\log r$, where r is the cross-ratio of the four points z_1, z_2, w_1, w_2, taken in an appropriate order.

Paper 4, Section II

15F Geometry

Suppose that D is the unit disc, with Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{1 - (x^2 + y^2)}.$$

[*Note that this is not a multiple of the Poincaré metric.*] Show that the diameters of D are, with appropriate parametrization, geodesics.

Show that distances between points in D are bounded, but areas of regions in D are unbounded.

Paper 2, Section I

2H Groups Rings and Modules

Give the definition of conjugacy classes in a group G. How many conjugacy classes are there in the symmetric group S_4 on four letters? Briefly justify your answer.

Paper 3, Section I

1H Groups Rings and Modules

Let A be the ring of integers $\mathbb{Z}$ or the polynomial ring $\mathbb{C}[X]$. In each case, give an example of an ideal I of A such that the quotient ring $R = A/I$ has a non-trivial idempotent (an element $x \in R$ with $x \neq 0, 1$ and $x^2 = x$) and a non-trivial nilpotent element (an element $x \in R$ with $x \neq 0$ and $x^n = 0$ for some positive integer n). Exhibit these elements and justify your answer.

Paper 4, Section I

2H Groups Rings and Modules

Let M be a free $\mathbb{Z}$-module generated by e_1 and e_2. Let a, b be two non-zero integers, and N be the submodule of M generated by $ae_1 + be_2$. Prove that the quotient module M/N is free if and only if a, b are coprime.

Paper 1, Section II

10H Groups Rings and Modules

Prove that the kernel of a group homomorphism $f : G \to H$ is a normal subgroup of the group G.

Show that the dihedral group D_8 of order 8 has a non-normal subgroup of order 2. Conclude that, for a group G, a normal subgroup of a normal subgroup of G is not necessarily a normal subgroup of G.

Paper 2, Section II

11H Groups Rings and Modules

For ideals I, J of a ring R, their *product* IJ is defined as the ideal of R generated by the elements of the form xy where $x \in I$ and $y \in J$.

(1) Prove that, if a prime ideal P of R contains IJ, then P contains either I or J.

(2) Give an example of R, I and J such that the two ideals IJ and $I \cap J$ are different from each other.

(3) Prove that there is a natural bijection between the prime ideals of R/IJ and the prime ideals of $R/(I \cap J)$.

Paper 3, Section II

11H Groups Rings and Modules

Let R be an integral domain and $R^\times$ its group of units. An element of $S = R \setminus (R^\times \cup \{0\})$ is *irreducible* if it is not a product of two elements in S. When R is Noetherian, show that every element of S is a product of finitely many irreducible elements of S.

Paper 4, Section II

11H Groups Rings and Modules

Let $V = (\mathbb{Z}/3\mathbb{Z})^2$, a 2-dimensional vector space over the field $\mathbb{Z}/3\mathbb{Z}$, and let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V$.

(1) List all 1-dimensional subspaces of V in terms of e_1, e_2. (For example, there is a subspace $\langle e_1 \rangle$ generated by e_1.)

(2) Consider the action of the matrix group

$$G = GL_2(\mathbb{Z}/3\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z}/3\mathbb{Z}, \; ad - bc \neq 0 \right\}$$

on the finite set X of all 1-dimensional subspaces of V. Describe the stabiliser group K of $\langle e_1 \rangle \in X$. What is the order of K? What is the order of G?

(3) Let $H \subset G$ be the subgroup of all elements of G which act trivially on X. Describe H, and prove that G/H is isomorphic to S_4, the symmetric group on four letters.

Paper 1, Section I

1F Linear Algebra

Suppose that V is the complex vector space of polynomials of degree at most $n-1$ in the variable z. Find the Jordan normal form for each of the linear transformations $\dfrac{d}{dz}$ and $z\dfrac{d}{dz}$ acting on V.

Paper 2, Section I

1F Linear Algebra

Suppose that ϕ is an endomorphism of a finite-dimensional complex vector space.

(i) Show that if λ is an eigenvalue of ϕ, then λ^2 is an eigenvalue of ϕ^2.

(ii) Show conversely that if μ is an eigenvalue of ϕ^2, then there is an eigenvalue λ of ϕ with $\lambda^2 = \mu$.

Paper 4, Section I

1F Linear Algebra

Define the notion of an inner product on a finite-dimensional real vector space V, and the notion of a self-adjoint linear map $\alpha : V \to V$.

Suppose that V is the space of real polynomials of degree at most n in a variable t. Show that

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\,dt$$

is an inner product on V, and that the map $\alpha : V \to V$:

$$\alpha(f)(t) = (1-t^2)f''(t) - 2tf'(t)$$

is self-adjoint.

Paper 1, Section II
9F Linear Algebra

Let V denote the vector space of $n \times n$ real matrices.

(1) Show that if $\psi(A, B) = \operatorname{tr}(AB^T)$, then ψ is a positive-definite symmetric bilinear form on V.

(2) Show that if $q(A) = \operatorname{tr}(A^2)$, then q is a quadratic form on V. Find its rank and signature.

[Hint: Consider symmetric and skew-symmetric matrices.]

Paper 2, Section II
10F Linear Algebra

(i) Show that two $n \times n$ complex matrices A, B are similar (i.e. there exists invertible P with $A = P^{-1}BP$) if and only if they represent the same linear map $\mathbb{C}^n \to \mathbb{C}^n$ with respect to different bases.

(ii) Explain the notion of Jordan normal form of a square complex matrix.

(iii) Show that any square complex matrix A is similar to its transpose.

(iv) If A is invertible, describe the Jordan normal form of A^{-1} in terms of that of A.

Justify your answers.

Paper 3, Section II
10F Linear Algebra

Suppose that V is a finite-dimensional vector space over $\mathbb{C}$, and that $\alpha : V \to V$ is a $\mathbb{C}$-linear map such that $\alpha^n = 1$ for some $n > 1$. Show that if V_1 is a subspace of V such that $\alpha(V_1) \subset V_1$, then there is a subspace V_2 of V such that $V = V_1 \oplus V_2$ and $\alpha(V_2) \subset V_2$.

[Hint: Show, for example by picking bases, that there is a linear map $\pi : V \to V_1$ with $\pi(x) = x$ for all $x \in V_1$. Then consider $\rho : V \to V_1$ with $\rho(y) = \frac{1}{n}\sum_{i=0}^{n-1} \alpha^i \pi \alpha^{-i}(y)$.]

Paper 4, Section II

10F Linear Algebra

(i) Show that the group $O_n(\mathbb{R})$ of orthogonal $n \times n$ real matrices has a normal subgroup $SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) \mid \det A = 1\}$.

(ii) Show that $O_n(\mathbb{R}) = SO_n(\mathbb{R}) \times \{\pm I_n\}$ if and only if n is odd.

(iii) Show that if n is even, then $O_n(\mathbb{R})$ is not the direct product of $SO_n(\mathbb{R})$ with any normal subgroup.

[*You may assume that the only elements of* $O_n(\mathbb{R})$ *that commute with all elements of* $O_n(\mathbb{R})$ *are* $\pm I_n$.]

Paper 3, Section I

9E Markov Chains

An intrepid tourist tries to ascend Springfield's famous infinite staircase on an icy day. When he takes a step with his right foot, he reaches the next stair with probability $1/2$, otherwise he falls down and instantly slides back to the bottom with probability $1/2$. Similarly, when he steps with his left foot, he reaches the next stair with probability $1/3$, or slides to the bottom with probability $2/3$. Assume that he always steps first with his right foot when he is at the bottom, and alternates feet as he ascends. Let X_n be his position after his nth step, so that $X_n = i$ when he is on the stair i, $i = 0, 1, 2, \ldots$, where 0 is the bottom stair.

(a) Specify the transition probabilities p_{ij} for the Markov chain $(X_n)_{n \geqslant 0}$ for any $i, j \geqslant 0$.

(b) Find the equilibrium probabilities π_i, for $i \geqslant 0$. [Hint: $\pi_0 = 5/9$.]

(c) Argue that the chain is irreducible and aperiodic and evaluate the limit

$$\lim_{n \to \infty} \mathbb{P}(X_n = i)$$

for each $i \geqslant 0$.

Paper 4, Section I

9E Markov Chains

Consider a Markov chain $(X_n)_{n \geqslant 0}$ with state space $\{a, b, c, d\}$ and transition probabilities given by the following table.

	a	b	c	d
a	$1/4$	$1/4$	$1/2$	0
b	0	$1/4$	0	$3/4$
c	$1/2$	0	$1/4$	$1/4$
d	0	$1/2$	0	$1/2$

By drawing an appropriate diagram, determine the communicating classes of the chain, and classify them as either open or closed. Compute the following transition and hitting probabilities:

- $\mathbb{P}(X_n = b \mid X_0 = d)$ for a fixed $n \geqslant 0$,
- $\mathbb{P}(X_n = c \text{ for some } n \geqslant 1 \mid X_0 = a)$.

Paper 1, Section II

20E Markov Chains

Let $(X_n)_{n \geqslant 0}$ be a Markov chain.

(a) What does it mean to say that a state i is positive recurrent? How is this property related to the equilibrium probability π_i? You do not need to give a full proof, but you should carefully state any theorems you use.

(b) What is a communicating class? Prove that if states i and j are in the same communicating class and i is positive recurrent then j is positive recurrent also.

A frog is in a pond with an infinite number of lily pads, numbered $1, 2, 3, \ldots$. She hops from pad to pad in the following manner: if she happens to be on pad i at a given time, she hops to one of pads $(1, 2, \ldots, i, i+1)$ with equal probability.

(c) Find the equilibrium distribution of the corresponding Markov chain.

(d) Now suppose the frog starts on pad k and stops when she returns to it. Show that the expected number of times the frog hops is $e(k-1)!$ where $e = 2.718 \ldots$. What is the expected number of times she will visit the lily pad $k+1$?

Paper 2, Section II

20E Markov Chains

Let $(X_n)_{n \geqslant 0}$ be a simple, symmetric random walk on the integers $\{\ldots, -1, 0, 1, \ldots\}$, with $X_0 = 0$ and $\mathbb{P}(X_{n+1} = i \pm 1 \mid X_n = i) = 1/2$. For each integer $a \geqslant 1$, let $T_a = \inf\{n \geqslant 0 : X_n = a\}$. Show that T_a is a stopping time.

Define a random variable Y_n by the rule

$$Y_n = \begin{cases} X_n & \text{if } n < T_a\,, \\ 2a - X_n & \text{if } n \geqslant T_a\,. \end{cases}$$

Show that $(Y_n)_{n \geqslant 0}$ is also a simple, symmetric random walk.

Let $M_n = \max_{0 \leqslant i \leqslant n} X_n$. Explain why $\{M_n \geqslant a\} = \{T_a \leqslant n\}$ for $a \geqslant 0$. By using the process $(Y_n)_{n \geqslant 0}$ constructed above, show that, for $a \geqslant 0$,

$$\mathbb{P}(M_n \geqslant a,\, X_n \leqslant a-1) = \mathbb{P}(X_n \geqslant a+1),$$

and thus

$$\mathbb{P}(M_n \geqslant a) = \mathbb{P}(X_n \geqslant a) + \mathbb{P}(X_n \geqslant a+1).$$

Hence compute

$$\mathbb{P}(M_n = a)$$

when a and n are positive integers with $n \geqslant a$. [Hint: if n is even, then X_n must be even, and if n is odd, then X_n must be odd.]

Paper 2, Section I

5A Methods

Consider the initial value problem

$$\mathcal{L}x(t) = f(t), \quad x(0) = 0, \quad \dot{x}(0) = 0, \quad t \geqslant 0,$$

where $\mathcal{L}$ is a second-order linear operator involving differentiation with respect to t. Explain briefly how to solve this by using a Green's function.

Now consider

$$\ddot{x}(t) = \begin{cases} a & 0 \leqslant t \leqslant T, \\ 0 & T < t < \infty, \end{cases}$$

where a is a constant, subject to the same initial conditions. Solve this using the Green's function, and explain how your answer is related to a problem in Newtonian dynamics.

Paper 3, Section I

7B Methods

Show that Laplace's equation $\nabla^2\phi = 0$ in polar coordinates (r, θ) has solutions proportional to $r^{\pm\alpha}\sin\alpha\theta$, $r^{\pm\alpha}\cos\alpha\theta$ for any constant α.

Find the function ϕ satisfying Laplace's equation in the region $a < r < b$, $0 < \theta < \pi$, where $\phi(a, \theta) = \sin^3\theta$, $\phi(b, \theta) = \phi(r, 0) = \phi(r, \pi) = 0$.

[*The Laplacian ∇^2 in polar coordinates is*

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$
]

Paper 4, Section I

5A Methods

(a) By considering strictly monotonic differentiable functions $\varphi(x)$, such that the zeros satisfy $\varphi(c) = 0$ but $\varphi'(c) \neq 0$, establish the formula

$$\int_{-\infty}^{\infty} f(x)\delta(\varphi(x))dx = \frac{f(c)}{|\varphi'(c)|}\,.$$

Hence show that for a general differentiable function with only such zeros, labelled by c,

$$\int_{-\infty}^{\infty} f(x)\delta(\varphi(x))\,dx = \sum_c \frac{f(c)}{|\varphi'(c)|}\,.$$

(b) Hence by changing to plane polar coordinates, or otherwise, evaluate,

$$I = \int_0^{\infty}\int_0^{\infty} (x^3 + y^2 x)\delta(x^2 + y^2 - 1)\,dydx.$$

Paper 1, Section II

14A Methods

(a) A function $f(t)$ is periodic with period 2π and has continuous derivatives up to and including the kth derivative. Show by integrating by parts that the Fourier coefficients of $f(t)$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt, \end{aligned}$$

decay at least as fast as $1/n^k$ as $n \to \infty$.

(b) Calculate the Fourier series of $f(t) = |\sin t|$ on $[0, 2\pi]$.

(c) Comment on the decay rate of your Fourier series.

Paper 2, Section II

16B Methods

Explain briefly the use of the method of characteristics to solve linear first-order partial differential equations.

Use the method to solve the problem

$$(x-y)\frac{\partial u}{\partial x}+(x+y)\frac{\partial u}{\partial y}=\alpha u,$$

where α is a constant, with initial condition $u(x,0)=x^2$, $x\geqslant 0$.

By considering your solution explain:

(i) why initial conditions cannot be specified on the whole x-axis;

(ii) why a single-valued solution in the entire plane is not possible if $\alpha\neq 2$.

Paper 3, Section II
15A Methods

(a) Put the equation

$$x\frac{d^2u}{dx^2} + \frac{du}{dx} + \lambda x u = 0, \qquad 0 \leqslant x \leqslant 1,$$

into Sturm–Liouville form.

(b) Suppose $u_n(x)$ are eigenfunctions such that $u_n(x)$ are bounded as x tends to zero and

$$x\frac{d^2u_n}{dx^2} + \frac{du_n}{dx} + \lambda_n x u_n = 0, \qquad 0 \leqslant x \leqslant 1.$$

Identify the weight function $w(x)$ and the most general boundary conditions on $u_n(x)$ which give the orthogonality relation

$$(\lambda_m - \lambda_n)\int_0^1 u_m(x)w(x)u_n(x)\,dx = 0.$$

(c) The equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0, \qquad x > 0,$$

has a solution $J_0(x)$ and a second solution which is not bounded at the origin. The zeros of $J_0(x)$ arranged in ascending order are $j_n, n = 1, 2, ...$ Given that $u_n(1) = 0$, show that the eigenvalues of the Sturm–Liouville problem in (b) are $\lambda = {j_n}^2, n = 1, 2, ...$

(d) Using the differential equations for $J_0(\alpha x)$ and $J_0(\beta x)$ and integration by parts, show that

$$\int_0^1 J_0(\alpha x)J_0(\beta x)x\,dx = \frac{\beta J_0(\alpha)J_0'(\beta) - \alpha J_0(\beta)J_0'(\alpha)}{\alpha^2 - \beta^2} \qquad (\alpha \neq \beta).$$

Paper 4, Section II

17B Methods

Defining the function $G_{f3}(\mathbf{r};\mathbf{r}_0) = -1/(4\pi|\mathbf{r}-\mathbf{r}_0|)$, prove Green's third identity for functions $u(\mathbf{r})$ satisfying Laplace's equation in a volume V with surface S, namely

$$u(\mathbf{r}_0) = \int_S \left(u\frac{\partial G_{f_3}}{\partial n} - \frac{\partial u}{\partial n} G_{f_3} \right) dS.$$

A solution is sought to the Neumann problem for $\nabla^2 u = 0$ in the half plane $z > 0$:

$$u = O(|\mathbf{x}|^{-a}), \quad \frac{\partial u}{\partial r} = O(|\mathbf{x}|^{-a-1}) \text{ as } |\mathbf{x}| \to \infty, \qquad \frac{\partial u}{\partial z} = p(x,y) \text{ on } z = 0,$$

where $a > 0$. It is assumed that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\,dx\,dy = 0$. Explain why this condition is necessary.

Construct an appropriate Green's function $G(\mathbf{r};\mathbf{r}_0)$ satisfying $\partial G/\partial z = 0$ at $z = 0$, using the method of images or otherwise. Hence find the solution in the form

$$u(x_0, y_0, z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) f(x - x_0, y - y_0, z_0)\,dx\,dy,$$

where f is to be determined.

Now let

$$p(x,y) = \begin{cases} x & |x|, |y| < a, \\ 0 & \text{otherwise.} \end{cases}$$

By expanding f in inverse powers of z_0, show that

$$u \to \frac{-2a^4 x_0}{3\pi z_0^3} \qquad \text{as} \quad z_0 \to \infty\ .$$

Paper 2, Section I

4H Metric and Topological Spaces

On the set $\mathbb{Q}$ of rational numbers, the 3-*adic metric* d_3 is defined as follows: for $x, y \in \mathbb{Q}$, define $d_3(x,x) = 0$ and $d_3(x,y) = 3^{-n}$, where n is the integer satisfying $x - y = 3^n u$ where u is a rational number whose denominator and numerator are both prime to 3.

(1) Show that this is indeed a metric on $\mathbb{Q}$.

(2) Show that in $(\mathbb{Q}, d_3)$, we have $3^n \to 0$ as $n \to \infty$ while $3^{-n} \not\to 0$ as $n \to \infty$. Let d be the usual metric $d(x,y) = |x - y|$ on $\mathbb{Q}$. Show that neither the identity map $(\mathbb{Q}, d) \to (\mathbb{Q}, d_3)$ nor its inverse is continuous.

Paper 3, Section I

3H Metric and Topological Spaces

Let X be a topological space and Y be a set. Let $p : X \to Y$ be a surjection. The *quotient topology* on Y is defined as follows: a subset $V \subset Y$ is open if and only if $p^{-1}(V)$ is open in X.

(1) Show that this does indeed define a topology on Y, and show that p is continuous when we endow Y with this topology.

(2) Let Z be another topological space and $f : Y \to Z$ be a map. Show that f is continuous if and only if $f \circ p : X \to Z$ is continuous.

Paper 1, Section II

12H Metric and Topological Spaces

Let $f : X \to Y$ and $g : Y \to X$ be continuous maps of topological spaces with $f \circ g = \mathrm{id}_Y$.

(1) Suppose that (i) Y is path-connected, and (ii) for every $y \in Y$, its inverse image $f^{-1}(y)$ is path-connected. Prove that X is path-connected.

(2) Prove the same statement when "path-connected" is everywhere replaced by "connected".

Paper 4, Section II

13H Metric and Topological Spaces

(1) Prove that if X is a compact topological space, then a closed subset Y of X endowed with the subspace topology is compact.

(2) Consider the following equivalence relation on $\mathbb{R}^2$:

$$(x_1, y_1) \sim (x_2, y_2) \iff (x_1 - x_2,\ y_1 - y_2) \in \mathbb{Z}^2.$$

Let $X = \mathbb{R}^2/\sim$ be the quotient space endowed with the quotient topology, and let $p : \mathbb{R}^2 \to X$ be the canonical surjection mapping each element to its equivalence class. Let $Z = \{(x, y) \in \mathbb{R}^2 \mid y = \sqrt{2}x\}$.

(i) Show that X is compact.

(ii) Assuming that $p(Z)$ is dense in X, show that $p|_Z : Z \to p(Z)$ is bijective but not homeomorphic.

Paper 1, Section I

6C Numerical Analysis

Obtain the Cholesky decompositions of

$$H_3 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}, \qquad H_4 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \lambda \end{pmatrix}.$$

What is the minimum value of λ for H_4 to be positive definite? Verify that if $\lambda = \frac{1}{7}$ then H_4 is positive definite.

Paper 4, Section I

8C Numerical Analysis

Suppose $x_0, x_1, \ldots, x_n \in [a, b] \subset \mathbf{R}$ are pointwise distinct and $f(x)$ is continuous on $[a, b]$. For $k = 1, 2, \ldots, n$ define

$$I_{0,k}(x) = \frac{f(x_0)(x_k - x) - f(x_k)(x_0 - x)}{x_k - x_0},$$

and for $k = 2, 3, \ldots, n$

$$I_{0,1,\ldots,k-2,k-1,k}(x) = \frac{I_{0,1,\ldots,k-2,k-1}(x)(x_k - x) - I_{0,1,\ldots,k-2,k}(x)(x_{k-1} - x)}{x_k - x_{k-1}}.$$

Show that $I_{0,1,\ldots,k-2,k-1,k}(x)$ is a polynomial of order k which interpolates $f(x)$ at $x_0, x_1, \ldots, x_k$.

Given $x_k = \{-1,\ 0,\ 2,\ 5\}$ and $f(x_k) = \{33,\ 5,\ 9,\ 1335\}$, determine the interpolating polynomial.

Paper 1, Section II

18C Numerical Analysis

Let

$$\langle f, g\rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x)\, g(x)\, dx\,,$$

be an inner product. The Hermite polynomials $H_n(x)$, $n = 0, 1, 2, \ldots$ are polynomials in x of degree n with leading term $2^n x^n$ which are orthogonal with respect to the inner product, with

$$\langle H_m, H_n\rangle = \begin{cases} \gamma_m > 0 & \text{if } m = n\,, \\ 0 & \text{otherwise,} \end{cases}$$

and $H_0(x) = 1$. Find a three-term recurrence relation which is satisfied by $H_n(x)$ and γ_n for $n = 1, 2, 3$. [*You may assume without proof that*

$$\langle 1, 1\rangle = \sqrt{\pi}\,, \quad \langle x, x\rangle = \tfrac{1}{2}\sqrt{\pi}\,, \quad \langle x^2, x^2\rangle = \tfrac{3}{4}\sqrt{\pi}\,, \quad \langle x^3, x^3\rangle = \tfrac{15}{8}\sqrt{\pi}\,.]$$

Next let $x_0, x_1, \ldots, x_k$ be the $k+1$ distinct zeros of $H_{k+1}(x)$ and for $i, j = 0, 1, \ldots, k$ define the Lagrangian polynomials

$$L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

associated with these points. Prove that $\langle L_i, L_j\rangle = 0$ if $i \neq j$.

Paper 2, Section II

19C Numerical Analysis

Consider the initial value problem for an autonomous differential equation

$$y'(t) = f(y(t)), \qquad y(0) = y_0 \text{ given,}$$

and its approximation on a grid of points $t_n = nh$, $n = 0, 1, 2, \ldots$. Writing $y_n = y(t_n)$, it is proposed to use one of two Runge–Kutta schemes defined by

$$y_{n+1} = y_n + \tfrac{1}{2}(k_1 + k_2),$$

where $k_1 = hf(y_n)$ and

$$k_2 = \begin{cases} hf(y_n + k_1) & \text{scheme I}\,, \\ hf(y_n + \tfrac{1}{2}(k_1 + k_2)) & \text{scheme II}\,. \end{cases}$$

What is the order of each scheme? Determine the A-stability of each scheme.

Paper 3, Section II

19C Numerical Analysis

Define the QR factorization of an $m \times n$ matrix A and explain how it can be used to solve the least squares problem of finding the $x^* \in \mathbf{R}^n$ which minimises $\|Ax - b\|$ where $b \in \mathbf{R}^m$, $m > n$, and the norm is the Euclidean one.

Define a Householder (reflection) transformation H and show that it is an orthogonal matrix.

Using a Householder reflection, solve the least squares problem for

$$A = \begin{pmatrix} 2 & 4 & 7 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix}, \qquad b = \begin{pmatrix} 9 \\ -7 \\ 3 \\ 1 \\ -1 \end{pmatrix},$$

giving both x^* and $\|Ax^* - b\|$.

Paper 1, Section I

8E Optimization

What is the maximal flow problem in a network?

Explain the Ford–Fulkerson algorithm. Why must this algorithm terminate if the initial flow is set to zero and all arc capacities are rational numbers?

Paper 2, Section I

9E Optimization

Consider the function ϕ defined by

$$\phi(b) = \inf\{x^2 + y^4 : \ x + 2\,y = b\}.$$

Use the Lagrangian sufficiency theorem to evaluate $\phi(3)$. Compute the derivative $\phi'(3)$.

Paper 3, Section II

21E Optimization

Let A be the $m \times n$ payoff matrix of a two-person, zero-sum game. What is Player I's optimization problem?

Write down a sufficient condition that a vector $p \in \mathbb{R}^m$ is an optimal mixed strategy for Player I in terms of the optimal mixed strategy of Player II and the value of the game. If $m = n$ and A is an invertible, symmetric matrix such that $A^{-1}e \geqslant 0$, where $e = (1, \dots, 1)^T \in \mathbb{R}^m$, show that the value of the game is $(e^T A^{-1} e)^{-1}$.

Consider the following game: Players I and II each have three cards labelled $1, 2$, and 3. Each player chooses one of her cards, independently of the other, and places it in the same envelope. If the sum of the numbers in the envelope is smaller than or equal to 4, then Player II pays Player I the sum (in £), and otherwise Player I pays Player II the sum. (For instance, if Player I chooses card 3 and Player II choose card 2, then Player I pays Player II £5.) What is the optimal strategy for each player?

Paper 4, Section II

20E Optimization

A factory produces three types of sugar, types X, Y, and Z, from three types of syrup, labelled A, B, and C. The following table contains the number of litres of syrup necessary to make each kilogram of sugar.

	X	Y	Z
A	3	2	1
B	2	3	2
C	4	1	2

For instance, one kilogram of type X sugar requires 3 litres of A, 2 litres of B, and 4 litres of C. The factory can sell each type of sugar for one pound per kilogram. Assume that the factory owner can use no more than 44 litres of A and 51 litres of B, but is required by law to use at least 12 litres of C. If her goal is to maximize profit, how many kilograms of each type of sugar should the factory produce?

Paper 3, Section I

8D Quantum Mechanics

Write down the commutation relations between the components of position $\mathbf{x}$ and momentum $\mathbf{p}$ for a particle in three dimensions.

A particle of mass m executes simple harmonic motion with Hamiltonian

$$H = \frac{1}{2m}\mathbf{p}^2 + \frac{m\omega^2}{2}\mathbf{x}^2,$$

and the orbital angular momentum operator is defined by

$$\mathbf{L} = \mathbf{x} \times \mathbf{p}.$$

Show that the components of $\mathbf{L}$ are observables commuting with H. Explain briefly why the components of $\mathbf{L}$ are not simultaneous observables. What are the implications for the labelling of states of the three-dimensional harmonic oscillator?

Paper 4, Section I

6D Quantum Mechanics

Determine the possible values of the energy of a particle free to move inside a cube of side a, confined there by a potential which is infinite outside and zero inside.

What is the degeneracy of the lowest-but-one energy level?

Paper 1, Section II

15D Quantum Mechanics

A particle of unit mass moves in one dimension in a potential

$$V = \frac{1}{2}\omega^2 x^2.$$

Show that the stationary solutions can be written in the form

$$\psi_n(x) = f_n(x)\exp(-\alpha x^2).$$

You should give the value of α and derive any restrictions on $f_n(x)$. Hence determine the possible energy eigenvalues E_n.

The particle has a wave function $\psi(x,t)$ which is even in x at $t=0$. Write down the general form for $\psi(x,0)$, using the fact that $f_n(x)$ is an even function of x only if n is even. Hence write down $\psi(x,t)$ and show that its probability density is periodic in time with period π/ω.

Paper 2, Section II

17D Quantum Mechanics

A particle of mass m moves in a one-dimensional potential defined by

$$V(x) = \begin{cases} \infty & \text{for } x<0, \\ 0 & \text{for } 0 \leqslant x \leqslant a, \\ V_0 & \text{for } a<x, \end{cases}$$

where a and V_0 are positive constants. Defining $c = [2m(V_0-E)]^{1/2}/\hbar$ and $k = (2mE)^{1/2}/\hbar$, show that for any allowed positive value E of the energy with $E < V_0$ then

$$c + k\cot ka = 0.$$

Find the minimum value of V_0 for this equation to have a solution.

Find the normalized wave function for the particle. Write down an expression for the expectation value of x in terms of two integrals, which you need not evaluate. Given that

$$\langle x\rangle = \frac{1}{2k}(ka - \tan ka),$$

discuss briefly the possibility of $\langle x\rangle$ being greater than a. [Hint: consider the graph of $-ka\cot ka$ against ka.]

Paper 3, Section II

16D Quantum Mechanics

A π^- (a particle of the same charge as the electron but 270 times more massive) is bound in the Coulomb potential of a proton. Assuming that the wave function has the form $ce^{-r/a}$, where c and a are constants, determine the normalized wave function of the lowest energy state of the π^-, assuming it to be an S-wave (i.e. the state with $l = 0$). (You should treat the proton as fixed in space.)

Calculate the probability of finding the π^- inside a sphere of radius R in terms of the ratio $\mu = R/a$, and show that this probability is given by $4\mu^3/3 + O(\mu^4)$ if μ is very small. Would the result be larger or smaller if the π^- were in a P-wave ($l = 1$) state? Justify your answer very briefly.

[Hint: in spherical polar coordinates,

$$\nabla^2\psi(\mathbf{r}) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}.\Big]$$

Paper 1, Section I

7E Statistics

Suppose $X_1, \ldots, X_n$ are independent $N(0, \sigma^2)$ random variables, where σ^2 is an unknown parameter. Explain carefully how to construct the uniformly most powerful test of size α for the hypothesis $H_0 : \sigma^2 = 1$ versus the alternative $H_1 : \sigma^2 > 1$.

Paper 2, Section I

8E Statistics

A washing powder manufacturer wants to determine the effectiveness of a television advertisement. Before the advertisement is shown, a pollster asks 100 randomly chosen people which of the three most popular washing powders, labelled A, B and C, they prefer. After the advertisement is shown, another 100 randomly chosen people (not the same as before) are asked the same question. The results are summarized below.

	A	B	C
before	36	47	17
after	44	33	23

Derive and carry out an appropriate test at the 5% significance level of the hypothesis that the advertisement has had no effect on people's preferences.

[*You may find the following table helpful:*

	χ_1^2	χ_2^2	χ_3^2	χ_4^2	χ_5^2	χ_6^2
95 percentile	3.84	5.99	7.82	9.49	11.07	12.59

.]

Paper 1, Section II

19E Statistics

Consider the the linear regression model

$$Y_i = \beta x_i + \epsilon_i,$$

where the numbers $x_1, \ldots, x_n$ are known, the independent random variables $\epsilon_1, \ldots, \epsilon_n$ have the $N(0, \sigma^2)$ distribution, and the parameters β and σ^2 are unknown. Find the maximum likelihood estimator for β.

State and prove the Gauss–Markov theorem in the context of this model.

Write down the distribution of an arbitrary linear estimator for β. Hence show that there exists a linear, unbiased estimator $\widehat{\beta}$ for β such that

$$\mathbb{E}_{\beta, \sigma^2}[(\widehat{\beta} - \beta)^4] \leqslant \mathbb{E}_{\beta, \sigma^2}[(\widetilde{\beta} - \beta)^4]$$

for all linear, unbiased estimators $\widetilde{\beta}$.

[Hint: If $Z \sim N(a, b^2)$ then $\mathbb{E}[(Z - a)^4] = 3 b^4$.]

Paper 3, Section II

20E Statistics

Let $X_1, \ldots, X_n$ be independent $\text{Exp}(\theta)$ random variables with unknown parameter θ. Find the maximum likelihood estimator $\hat{\theta}$ of θ, and state the distribution of $n/\hat{\theta}$. Show that $\theta/\hat{\theta}$ has the $\Gamma(n, n)$ distribution. Find the $100\,(1-\alpha)\%$ confidence interval for θ of the form $[0, C\,\hat{\theta}]$ for a constant $C > 0$ depending on α.

Now, taking a Bayesian point of view, suppose your prior distribution for the parameter θ is $\Gamma(k, \lambda)$. Show that your Bayesian point estimator $\hat{\theta}_B$ of θ for the loss function $L(\theta, a) = (\theta - a)^2$ is given by

$$\hat{\theta}_B = \frac{n + k}{\lambda + \sum_i X_i}.$$

Find a constant $C_B > 0$ depending on α such that the posterior probability that $\theta \leqslant C_B\,\hat{\theta}_B$ is equal to $1 - \alpha$.

[*The density of the* $\Gamma(k, \lambda)$ *distribution is* $f(x;\, k, \lambda) = \lambda^k x^{k-1} e^{-\lambda x}/\Gamma(k)$, *for* $x > 0$.]

Paper 4, Section II

19E Statistics

Consider a collection $X_1, \ldots, X_n$ of independent random variables with common density function $f(x;\theta)$ depending on a real parameter θ. What does it mean to say T is a sufficient statistic for θ? Prove that if the joint density of $X_1, \ldots, X_n$ satisfies the factorisation criterion for a statistic T, then T is sufficient for θ.

Let each X_i be uniformly distributed on $[-\sqrt{\theta}, \sqrt{\theta}\,]$. Find a two-dimensional sufficient statistic $T = (T_1, T_2)$. Using the fact that $\hat{\theta} = 3X_1^2$ is an unbiased estimator of θ, or otherwise, find an unbiased estimator of θ which is a function of T and has smaller variance than $\hat{\theta}$. Clearly state any results you use.

Paper 1, Section I

4D Variational Principles

(a) Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be convex and strictly convex.

(b) State a necessary and sufficient first-order condition for strict convexity of $f \in C^1(\mathbb{R}^n)$, and give, with proof, an example of a function which is strictly convex but with second derivative which is not everywhere strictly positive.

Paper 3, Section I

6D Variational Principles

Derive the Euler-Lagrange equation for the function $u(x, y)$ which gives a stationary value of

$$I = \int_{\mathcal{D}} L\Big(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big)\, dxdy,$$

where $\mathcal{D}$ is a bounded domain in the (x, y) plane, with u fixed on the boundary $\partial\mathcal{D}$.

Find the equation satisfied by the function u which gives a stationary value of

$$I = \int_{\mathcal{D}} \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2\Big]\, dxdy,$$

with u given on $\partial\mathcal{D}$.

Paper 2, Section II

15D Variational Principles

Describe briefly the method of Lagrange multipliers for finding the stationary points of a function $f(x, y)$ subject to a constraint $\phi(x, y) = 0$.

A tent manufacturer wants to maximize the volume of a new design of tent, subject only to a constant weight (which is directly proportional to the amount of fabric used). The models considered have either equilateral-triangular or semi-circular vertical cross–section, with vertical planar ends in both cases and with floors of the same fabric. Which shape maximizes the volume for a given area A of fabric?

[Hint: $(2\pi)^{-1/2}3^{-3/4}(2 + \pi) < 1$.]

Paper 4, Section II
16D Variational Principles

A function $\theta(\phi)$ with given values of $\theta(\phi_1)$ and $\theta(\phi_2)$ makes the integral

$$I = \int_{\phi_1}^{\phi_2} \mathcal{L}(\theta, \theta')\, d\phi$$

stationary with respect to small variations of θ which vanish at ϕ_1 and ϕ_2. Show that $\theta(\phi)$ satisfies the first integral of the Euler–Lagrange equation,

$$\mathcal{L}(\theta, \theta') - \theta'(\partial \mathcal{L}/\partial \theta') = C,$$

for some constant C. You may state the Euler–Lagrange equation without proof.

It is desired to tow an iceberg across open ocean from a point on the Antarctic coast (longitude ϕ_1) to a place in Australia (longitude ϕ_2), to provide fresh water for irrigation. The iceberg will melt at a rate proportional to the difference between its temperature (the constant T_0, measured in degrees Celsius and therefore negative) and the sea temperature $T(\theta) > T_0$, where θ is the colatitude (the usual spherical polar coordinate θ). Assume that the iceberg is towed at a constant speed along a path $\theta(\phi)$, where ϕ is the longitude. Given that the infinitesimal arc length on the unit sphere is $(d\theta^2 + \sin^2\theta\; d\phi^2)^{1/2}$, show that the total ice melted along the path from ϕ_1 to ϕ_2 is proportional to

$$I = \int_{\phi_1}^{\phi_2} (T(\theta) - T_0)(\theta'^2 + \sin^2\theta)^{1/2}\, d\phi.$$

Now suppose that in the relevant latitudes, the sea temperature may be approximated by $T(\theta) = T_0(1+3\tan\theta)$. (*Note that* $(1+3\tan\theta)$ *is negative in the relevant latitudes.*) Show that any smooth path $\theta(\phi)$ which minimizes the total ice melted must satisfy

$$\theta'^2 = \sin^2\theta\,(\tfrac{1}{4}k^2\tan^2\theta\sin^2\theta - 1),$$

and hence that

$$\sin^2\theta = \frac{2}{1-(1+k^2)^{1/2}\sin 2(\phi-\phi_0)},$$

where k and ϕ_0 are constants.

[Hint:

$$\int \frac{dx}{x(\alpha^2x^4 + x^2 - 1)^{1/2}} = \frac{1}{2}\arcsin\left[\frac{x^2-2}{x^2(1+4\alpha^2)^{1/2}}\right]. \,]$$