UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 5 June, 2009 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, ..., H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Linear Algebra

Show that every endomorphism of a finite-dimensional vector space satisfies some polynomial, and define the *minimal polynomial* of such an endomorphism.

Give a linear transformation of an eight-dimensional complex vector space which has minimal polynomial $x^2(x-1)^3$.

2F Groups, Rings and Modules

Let M be a module over an integral domain R. An element $m \in M$ is said to be torsion if there exists a nonzero $r \in R$ with $rm = 0$; M is said to be torsion-free if its only torsion element is 0. Show that there exists a unique submodule N of M such that (a) all elements of N are torsion and (b) the quotient module M/N is torsion-free.

3E Analysis II

Let $(s_n)_{n=1}^{\infty}$ be a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and let $s : \mathbb{R} \to \mathbb{R}$ be another continuous function. What does it mean to say that $s_n \to s$ uniformly? Give examples (without proof) of a sequence (s_n) of nonzero functions which converges to 0 uniformly, and of a sequence which converges to 0 pointwise but *not* uniformly. Show that if $s_n \to s$ uniformly then

$$\int_{-1}^{1} s_n(x)\,dx \to \int_{-1}^{1} s(x)\,dx.$$

Give an example of a continuous function $s : \mathbb{R} \to \mathbb{R}$ with $s(x) \geqslant 0$ for all x, $s(x) \to 0$ as $|x| \to \infty$ but for which $\int_{-\infty}^{\infty} s(x)\,dx$ does not converge. For each positive integer n define $s_n(x)$ to be equal to $s(x)$ if $|x| \leqslant n$, and to be $s(n)\min(1, ||x|-n|^{-2})$ for $|x| > n$. Show that the functions s_n are continuous, tend uniformly to s, and furthermore that $\int_{-\infty}^{\infty} s_n(x)\,dx$ exists and is finite for all n.

4E Complex Analysis

State Rouché's Theorem. How many complex numbers z are there with $|z| \leqslant 1$ and $2z = \sin z$?

5B Mathematical Methods

Describe briefly the method of Lagrange multipliers for finding the stationary points of a function $f(x,y)$ subject to the constraint $g(x,y)=0$.

Show that at a stationary point (a,b)

$$\begin{vmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial g}{\partial x}(a,b) \\ \frac{\partial f}{\partial y}(a,b) & \frac{\partial g}{\partial y}(a,b) \end{vmatrix} = 0 .$$

Find the maximum distance from the origin to the curve

$$x^2+y^2+xy-4=0 .$$

6B Quantum Mechanics

The wavefunction of a Gaussian wavepacket for a particle of mass m moving in one dimension is

$$\psi(x,t)=\frac{1}{\pi^{1/4}}\sqrt{\frac{1}{1+i\hbar t/m}}\exp\left(-\frac{x^2}{2(1+i\hbar t/m)}\right) .$$

Show that $\psi(x,t)$ satisfies the appropriate time-dependent Schrödinger equation.

Show that $\psi(x,t)$ is normalized to unity and calculate the uncertainty in measurement of the particle position, $\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}$.

Is $\psi(x,t)$ a stationary state? Give a reason for your answer.

$\Big[$*You may assume that* $\int_{-\infty}^{\infty} e^{-\lambda x^2}\,dx=\sqrt{\frac{\pi}{\lambda}}$.$\Big]$

7A Electromagnetism

State the relationship between the induced EMF V in a loop and the flux Φ through it. State the force law for a current-carrying wire in a magnetic field $\mathbf{B}$.

A rectangular loop of wire with mass m, width w, vertical length l, and resistance R falls out of a magnetic field under the influence of gravity. The magnetic field is $\mathbf{B}=B\hat{\mathbf{x}}$ for $z\geqslant 0$ and $\mathbf{B}=0$ for $z<0$, where B is constant. Suppose the loop lies in the (y,z) plane, with its top initially at $z=z_0<l$. Find the equation of motion for the loop and its terminal velocity, assuming that the loop continues to intersect the plane $z=0$.

8C Numerical Analysis

Suppose that $w(x) > 0$ for all $x \in (a,b)$. The weights $b_1, ..., b_n$ and nodes $x_1, ..., x_n$ are chosen so that the Gaussian quadrature formula

$$\int_a^b w(x)f(x)dx \sim \sum_{k=1}^{n} b_k f(x_k)$$

is exact for every polynomial of degree $2n-1$. Show that the $b_i, i = 1, ..., n$ are all positive.

When $w(x) = 1 + x^2$, $a = -1$ and $b = 1$, the first three underlying orthogonal polynomials are $p_0(x) = 1, p_1(x) = x$, and $p_2(x) = x^2 - 2/5$. Find x_1, x_2 and b_1, b_2 when $n = 2$.

9H Markov Chains

In chess, a bishop is allowed to move only in straight diagonal lines. Thus if the bishop stands on the square marked A in the diagram, it is able in one move to reach any of the squares marked with an asterisk. Suppose that the bishop moves at random around the chess board, choosing at each move with equal probability from the squares it can reach, the square chosen being independent of all previous choices. The bishop starts at the bottom left-hand corner of the board.

If X_n is the position of the bishop at time n, show that $(X_n)_{n\geqslant 0}$ is a reversible Markov chain, whose statespace you should specify. Find the invariant distribution of this Markov chain.

What is the expected number of moves the bishop will make before first returning to its starting square?

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SECTION II

10G Linear Algebra

What does it mean to say two real symmetric bilinear forms A and B on a vector space V are *congruent* ?

State and prove Sylvester's law of inertia, and deduce that the rank and signature determine the congruence class of a real symmetric bilinear form. [*You may use without proof a result on diagonalisability of real symmetric matrices, provided it is clearly stated.*]

How many congruence classes of symmetric bilinear forms on a real n-dimensional vector space are there? Such a form ψ defines a family of subsets $\{x \in \mathbb{R}^n \mid \psi(x,x) = t\}$, for $t \in \mathbb{R}$. For how many of the congruence classes are these associated subsets all bounded subsets of $\mathbb{R}^n$? Is the quadric surface

$$\{3x^2 + 6y^2 + 5z^2 + 4xy + 2xz + 8yz = 1\}$$

a bounded or unbounded subset of $\mathbb{R}^3$? Justify your answers.

11F Groups, Rings and Modules

Let R be a principal ideal domain. Prove that any submodule of a finitely-generated free module over R is free.

An R-module P is said to be projective if, whenever we have module homomorphisms $f : M \to N$ and $g : P \to N$ with f surjective, there exists a homomorphism $h \colon P \to M$ with $f \circ h = g$. Show that any free module (over an arbitrary ring) is projective. Show also that a finitely-generated projective module over a principal ideal domain is free.

12G Geometry

Let $U \subset \mathbb{R}^2$ be an open set. Let $\Sigma \subset \mathbb{R}^3$ be a surface locally given as the graph of an infinitely-differentiable function $f : U \to \mathbb{R}$. Compute the Gaussian curvature of Σ in terms of f.

Deduce that if $\widehat{\Sigma} \subset \mathbb{R}^3$ is a compact surface without boundary, its Gaussian curvature is not everywhere negative.

Give, with brief justification, a compact surface $\widehat{\Sigma} \subset \mathbb{R}^3$ without boundary whose Gaussian curvature must change sign.

13E Analysis II

Let (X, d) be a metric space with at least two points. If $f : X \to \mathbb{R}$ is a function, write

$$\mathrm{Lip}(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} + \sup_{z} |f(z)|,$$

provided that this supremum is finite. Let $\mathrm{Lip}(X) = \{f : \mathrm{Lip}(f) \text{ is defined}\}$. Show that $\mathrm{Lip}(X)$ is a vector space over $\mathbb{R}$, and that Lip is a norm on it.

Now let $X = \mathbb{R}$. Suppose that $(f_i)_{i=1}^{\infty}$ is a sequence of functions with $\mathrm{Lip}(f_i) \leqslant 1$ and with the property that the sequence $f_i(q)$ converges as $i \to \infty$ for every rational number q. Show that the f_i converge pointwise to a function f satisfying $\mathrm{Lip}(f) \leqslant 1$.

Suppose now that $(f_i)_{i=1}^{\infty}$ are any functions with $\mathrm{Lip}(f_i) \leqslant 1$. Show that there is a subsequence $f_{i_1}, f_{i_2}, \ldots$ which converges pointwise to a function f with $\mathrm{Lip}(f) \leqslant 1$.

14F Metric and Topological Spaces

A nonempty subset A of a topological space X is called *irreducible* if, whenever we have open sets U and V such that $U \cap A$ and $V \cap A$ are nonempty, then we also have $U \cap V \cap A \neq \emptyset$. Show that the closure of an irreducible set is irreducible, and deduce that the closure of any singleton set $\{x\}$ is irreducible.

X is said to be a *sober* topological space if, for any irreducible closed $A \subseteq X$, there is a unique $x \in X$ such that $A = \overline{\{x\}}$. Show that any Hausdorff space is sober, but that an infinite set with the cofinite topology is not sober.

Given an arbitrary topological space $(X, \mathcal{T})$, let $\widehat{X}$ denote the set of all irreducible closed subsets of X, and for each $U \in \mathcal{T}$ let

$$\widehat{U} = \{A \in \widehat{X} \mid U \cap A \neq \emptyset\}\ .$$

Show that the sets $\{\widehat{U} \mid U \in \mathcal{T}\}$ form a topology $\widehat{\mathcal{T}}$ on $\widehat{X}$, and that the mapping $U \mapsto \widehat{U}$ is a bijection from $\mathcal{T}$ to $\widehat{\mathcal{T}}$. Deduce that $(\widehat{X}, \widehat{\mathcal{T}})$ is sober. [*Hint: consider the complement of an irreducible closed subset of* $\widehat{X}$.]

15D Complex Methods

The function $u(x, y)$ satisfies Laplace's equation in the half-space $y \geqslant 0$, together with boundary conditions

$$u(x, y) \to 0 \text{ as } y \to \infty \text{ for all } x \ ,$$
$$u(x, 0) = u_0(x), \text{ where } x\, u_0(x) \to 0 \text{ as } |x| \to \infty.$$

Using Fourier transforms, show that

$$u(x, y) = \int_{-\infty}^{\infty} u_0(t) v(x - t, y) dt \ ,$$

where

$$v(x, y) = \frac{y}{\pi(x^2 + y^2)} \ .$$

Suppose that $u_0(x) = (x^2 + a^2)^{-1}$. Using contour integration and the convolution theorem, or otherwise, show that

$$u(x, y) = \frac{y + a}{a\,[x^2 + (y + a)^2]} \ .$$

[*You may assume the convolution theorem of Fourier transforms, i.e. that if* $\tilde{f}(k), \tilde{g}(k)$ *are the Fourier transforms of two functions* $f(x), g(x)$, *then* $\tilde{f}(k)\tilde{g}(k)$ *is the Fourier transform of* $\int_{-\infty}^{\infty} f(t)g(x-t)dt$.]

16A Methods

Suppose that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of

$$\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = 0 \ ,$$

with $y_1(0) = 0$ and $y_2(1) = 0$. Show that the Green's function $G(x, \xi)$ for the interval $0 \leqslant x, \xi \leqslant 1$ and with $G(0, \xi) = G(1, \xi) = 0$ can be written in the form

$$G(x,\xi) = \begin{cases} y_1(x)y_2(\xi)/W(\xi); & 0 < x < \xi, \\ y_2(x)y_1(\xi)/W(\xi); & \xi < x < 1, \end{cases}$$

where $W(x) = W[y_1(x), y_2(x)]$ is the Wronskian of $y_1(x)$ and $y_2(x)$.

Use this result to find the Green's function $G(x, \xi)$ that satisfies

$$\frac{d^2G}{dx^2} + 3\frac{dG}{dx} + 2G = \delta(x - \xi) \ ,$$

in the interval $0 \leqslant x, \xi \leqslant 1$ and with $G(0, \xi) = G(1, \xi) = 0$. Hence obtain an integral expression for the solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \begin{cases} 0; & 0 < x < x_0, \\ 2; & x_0 < x < 1, \end{cases}$$

for the case $x < x_0$.

17C Special Relativity

A star moves with speed v in the x-direction in a reference frame S. When viewed in its rest frame S' it emits a photon of frequency ν' which propagates along a line making an angle θ' with the x'-axis. Write down the components of the four-momentum of the photon in S'. As seen in S, the photon moves along a line that makes an angle θ with the x-axis and has frequency ν. Using a Lorentz transformation, write down the relationship between the components of the four-momentum of the photon in S' to those in S and show that

$$\cos\theta = \frac{\cos\theta' + v/c}{1 + v\cos\theta'/c}.$$

As viewed in S', the star emits two photons with frequency ν' in opposite directions with $\theta' = \pi/2$ and $\theta' = -\pi/2$, respectively. Show that an observer in S records them as having a combined momentum p directed along the x-axis, where

$$p = \frac{Ev}{c^2\sqrt{1 - v^2/c^2}}$$

and where E is the combined energy of the photons as seen in S'. How is this momentum loss from the star consistent with its maintaining a constant speed as viewed in S?

18D Fluid Dynamics

An inviscid incompressible fluid occupies a rectangular tank with vertical sides at $x = 0, a$ and $y = 0, b$ and a horizontal bottom at $z = -h$. The undisturbed free surface is at $z = 0$.

(i) Write down the equations and boundary conditions governing small amplitude free oscillations of the fluid, neglecting surface tension, and show that the frequencies ω of such oscillations are given by

$$\frac{\omega^2}{g} = k \tanh kh, \text{ where } k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \qquad (1)$$

for non-negative integers m, n, which cannot both be zero.

(ii) The free surface is now acted on by a small external pressure

$$p = \epsilon\rho g h \sin\Omega t \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b},$$

where $\epsilon \ll 1$. Calculate the corresponding oscillation of the free surface when Ω is not equal to the quantity ω given by (1).

Why does your solution break down as $\Omega \to \omega$?

19H Statistics

What is a *sufficient statistic*? State the factorization criterion for a statistic to be sufficient.

Suppose that $X_1, \ldots, X_n$ are independent random variables uniformly distributed over $[a, b]$, where the parameters $a < b$ are not known, and $n \geqslant 2$. Find a sufficient statistic for the parameter $\theta \equiv (a, b)$ based on the sample $X_1, \ldots, X_n$. Based on your sufficient statistic, derive an unbiased estimator of θ.

20H Optimization

In a pure exchange economy, there are J agents, and d goods. Agent j initially holds an endowment $x_j \in \mathbb{R}^d$ of the d different goods, $j = 1, \ldots, J$. Agent j has preferences given by a concave utility function $U_j : \mathbb{R}^d \to \mathbb{R}$ which is strictly increasing in each of its arguments, and is twice continuously differentiable. Thus agent j prefers $y \in \mathbb{R}^d$ to $x \in \mathbb{R}^d$ if and only if $U_j(y) \geqslant U_j(x)$.

The agents meet and engage in mutually beneficial trades. Thus if agent i holding z_i meets agent j holding z_j, then the amounts z_i' held by agent i and z_j' held by agent j after trading must satisfy $U_i(z_i') \geqslant U_i(z_i)$, $U_j(z_j') \geqslant U_j(z_j)$, and $z_i' + z_j' = z_i + z_j$. Meeting and trading continues until, finally, agent j holds $y_j \in \mathbb{R}^d$, where

$$\sum_j x_j = \sum_j y_j,$$

and there are no further mutually beneficial trades available to any pair of agents. Prove that there must exist a vector $v \in \mathbb{R}^d$ and positive scalars $\lambda_1, \ldots, \lambda_J$ such that

$$\nabla U_j(y_j) = \lambda_j v$$

for all j. Show that for some positive $a_1, \ldots, a_J$ the final allocations y_j are what would be achieved by a social planner, whose objective is to obtain

$$\max \sum_j a_j U_j(y_j) \quad \text{subject to} \quad \sum_j y_j = \sum_j x_j.$$

END OF PAPER