UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 4 June, 2009 9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, ..., H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Groups, Rings and Modules

Let F be a field. Show that the polynomial ring $F[X]$ is a principal ideal domain. Give, with justification, an example of an ideal in $F[X, Y]$ which is not principal.

2G Geometry

Write down the equations for geodesic curves on a surface. Use these to describe all the geodesics on a circular cylinder, and draw a picture illustrating your answer.

3E Analysis II

What is meant by a *norm* on $\mathbb{R}^n$? For $\mathbf{x} \in \mathbb{R}^n$ write

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|,$$

$$\|\mathbf{x}\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}.$$

Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms. [*You may assume the Cauchy-Schwarz inequality.*]

Find the smallest constant C_n such that $\|x\|_1 \leqslant C_n \|x\|_2$ for all $x \in \mathbb{R}^n$, and also the smallest constant C'_n such that $\|x\|_2 \leqslant C'_n \|x\|_1$ for all $x \in \mathbb{R}^n$.

4F Metric and Topological Spaces

Are the following statements true or false? Give brief justifications for your answers.

(i) If X is a connected open subset of $\mathbb{R}^n$ for some n, then X is path-connected.

(ii) A cartesian product of two connected spaces is connected.

(iii) If X is a Hausdorff space and the only connected subsets of X are singletons $\{x\}$, then X is discrete.

5D Complex Methods

Use the residue calculus to evaluate

$$\text{(i)} \quad \oint_C z e^{1/z} dz \quad \text{and} \quad \text{(ii)} \quad \oint_C \frac{z\,dz}{1 - 4z^2}\,,$$

where C is the circle $|z| = 1$.

6A Methods

The Fourier transform $\tilde{f}(\omega)$ of a suitable function $f(t)$ is defined as $\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$. Consider the function $h(t) = e^{\alpha t}$ for $t > 0$, and zero otherwise. Show that

$$\tilde{h}(\omega) = \frac{1}{i\omega - \alpha},$$

provided $\Re(\alpha) < 0$.

The angle $\theta(t)$ of a forced, damped pendulum satisfies

$$\ddot{\theta} + 2\dot{\theta} + 5\theta = e^{-4t},$$

with initial conditions $\theta(0) = \dot{\theta}(0) = 0$. Show that the transfer function for this system is

$$\tilde{R}(\omega) = \frac{1}{4i}\left[\frac{1}{(i\omega + 1 - 2i)} - \frac{1}{(i\omega + 1 + 2i)}\right].$$

7B Quantum Mechanics

The motion of a particle in one dimension is described by the time-independent hermitian Hamiltonian operator H whose normalized eigenstates $\psi_n(x)$, $n = 0, 1, 2, \ldots,$ satisfy the Schrödinger equation

$$H\psi_n = E_n\psi_n\ ,$$

with $E_0 < E_1 < E_2 < \cdots < E_n < \cdots$. Show that

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = \delta_{mn}\ .$$

The particle is in a state represented by the wavefunction $\Psi(x,t)$ which, at time $t = 0$, is given by

$$\Psi(x,0) = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^{n+1} \psi_n(x)\ .$$

Write down an expression for $\Psi(x,t)$ and show that it is normalized to unity.

Derive an expression for the expectation value of the energy for this state and show that it is independent of time.

Calculate the probability that the particle has energy E_m for a given integer $m \geqslant 0$, and show that this also is time-independent.

8H Statistics

In a demographic study, researchers gather data on the gender of children in families with more than two children. For each of the four possible outcomes GG, GB, BG, BB of the first two children in the family, they find 50 families which started with that pair, and record the gender of the third child of the family. This produces the following table of counts:

First two children	Third child B	Third child G
GG	16	34
GB	28	22
BG	25	25
BB	31	19

In view of this, is the hypothesis that the gender of the third child is independent of the genders of the first two children rejected at the 5% level?

[*Hint: the* 95% *point of a* χ^2_3 *distribution is* 7.8147*, and the* 95% *point of a* χ^2_4 *distribution is* 9.4877.]

9H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a simple random walk on the integers: the random variables $\xi_n \equiv X_n - X_{n-1}$ are independent, with distribution

$$P(\xi = 1) = p, \quad P(\xi = -1) = q,$$

where $0 < p < 1$, and $q = 1-p$. Consider the hitting time $\tau = \inf\{n : X_n = 0 \text{ or } X_n = N\}$, where $N > 1$ is a given integer. For fixed $s \in (0,1)$ define $\xi_k = E[s^\tau : X_\tau = 0 | X_0 = k]$ for $k = 0, \ldots, N$. Show that the ξ_k satisfy a second-order difference equation, and hence find them.

SECTION II

10G Linear Algebra

For each of the following, provide a proof or counterexample.

(1) If A, B are complex $n \times n$ matrices and $AB = BA$, then A and B have a common eigenvector.

(2) If A, B are complex $n \times n$ matrices and $AB = BA$, then A and B have a common eigenvalue.

(3) If A, B are complex $n \times n$ matrices and $(AB)^n = 0$ then $(BA)^n = 0$.

(4) If $T : V \to V$ is an endomorphism of a finite-dimensional vector space V and λ is an eigenvalue of T, then the dimension of $\{v \in V \mid (T - \lambda I)v = 0\}$ equals the multiplicity of λ as a root of the minimal polynomial of T.

(5) If $T : V \to V$ is an endomorphism of a finite-dimensional complex vector space V, λ is an eigenvalue of T, and $W_i = \{v \in V \mid (T - \lambda I)^i(v) = 0\}$, then $W_c = W_{c+1}$ where c is the multiplicity of λ as a root of the minimal polynomial of T.

11F Groups, Rings and Modules

Let S be a multiplicatively closed subset of a ring R, and let I be an ideal of R which is maximal among ideals disjoint from S. Show that I is prime.

If R is an integral domain, explain briefly how one may construct a field F together with an injective ring homomorphism $R \to F$.

Deduce that if R is an arbitrary ring, I an ideal of R, and S a multiplicatively closed subset disjoint from I, then there exists a ring homomorphism $f : R \to F$, where F is a field, such that $f(x) = 0$ for all $x \in I$ and $f(y) \neq 0$ for all $y \in S$.

[*You may assume that if T is a multiplicatively closed subset of a ring, and $0 \notin T$, then there exists an ideal which is maximal among ideals disjoint from T.*]

12G Geometry

Consider a tessellation of the two-dimensional sphere, that is to say a decomposition of the sphere into polygons each of which has at least three sides. Let E, V and F denote the numbers of edges, vertices and faces in the tessellation, respectively. State Euler's formula. Prove that $2E \geqslant 3F$. Deduce that not all the vertices of the tessellation have valence $\geqslant 6$.

By considering the plane $\{z = 1\} \subset \mathbb{R}^3$, or otherwise, deduce the following: if Σ is a finite set of straight lines in the plane $\mathbb{R}^2$ with the property that every intersection point of two lines is an intersection point of at least three, then all the lines in Σ meet at a single point.

13E Analysis II

What does it mean for a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ of several variables to be differentiable at a point $\mathbf{x}$? State and prove the chain rule for functions of several variables. For each of the following two functions from $\mathbb{R}^2$ to $\mathbb{R}$, give with proof the set of points at which it is differentiable:

$$g_1(x,y) = \begin{cases} (x^2 - y^2)\sin\frac{1}{x^2-y^2} & \text{if } x \neq \pm y \\ 0 & \text{otherwise;} \end{cases}$$

$$g_2(x,y) = \begin{cases} (x^2 + y^2)\sin\frac{1}{x^2+y^2} & \text{if at least one of } x, y \text{ is not } 0 \\ 0 & \text{if } x = y = 0. \end{cases}$$

14E Complex Analysis

For each positive real number R write $B_R = \{z \in \mathbb{C} : |z| \leqslant R\}$. If F is holomorphic on some open set containing B_R, we define

$$J(F, R) = \frac{1}{2\pi} \int_0^{2\pi} \log |F(Re^{i\theta})| \, d\theta.$$

1. If F_1, F_2 are both holomorphic on some open set containing B_R, show that $J(F_1F_2, R) = J(F_1, R) + J(F_2, R)$.

2. Suppose that $F(0) = 1$ and that F does not vanish on some open set containing B_R. By showing that there is a holomorphic branch of logarithm of F and then considering $z^{-1} \log F(z)$, prove that $J(F, R) = 0$.

3. Suppose that $|w| < R$. Prove that the function $\psi_{W,R}(z) = R(z - w)/(R^2 - \overline{w}z)$ has modulus 1 on $|z| = R$ and hence that it satisfies $J(\psi_{W,R}, R) = 0$.

Suppose now that $F : \mathbb{C} \to \mathbb{C}$ is holomorphic and not identically zero, and let R be such that no zeros of F satisfy $|z| = R$. Briefly explain why there are only finitely many zeros of F in B_R and, assuming these are listed with the correct multiplicity, derive a formula for $J(F, R)$ in terms of the zeros, R, and $F(0)$.

Suppose that F has a zero at every lattice point (point with integer coordinates) except for $(0, 0)$. Show that there is a constant $c > 0$ such that $|F(z_i)| > e^{c|z_i|^2}$ for a sequence $z_1, z_2, \ldots$ of complex numbers tending to infinity.

15A Methods

A function $g(r)$ is chosen to make the integral

$$\int_a^b f(r, g, g')dr$$

stationary, subject to given values of $g(a)$ and $g(b)$. Find the Euler–Lagrange equation for $g(r)$.

In a certain three-dimensional electrostatics problem the potential ϕ depends only on the radial coordinate r, and the energy functional of ϕ is

$$\mathcal{E}[\phi] = 2\pi \int_{R_1}^{R_2} \left[\frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + \frac{1}{2\lambda^2}\phi^2 \right] r^2 dr \ ,$$

where λ is a parameter. Show that the Euler–Lagrange equation associated with minimizing the energy $\mathcal{E}$ is equivalent to

$$\frac{1}{r}\frac{d^2\left(r\phi\right)}{dr^2} - \frac{1}{\lambda^2}\phi = 0 \ . \qquad (1)$$

Find the general solution of this equation, and the solution for the region $R_1 \leqslant r \leqslant R_2$ which satisfies $\phi(R_1) = \phi_1$ and $\phi(R_2) = 0$.

Consider an annular region in two dimensions, where the potential is a function of the radial coordinate r only. Write down the equivalent expression for the energy functional $\mathcal{E}$ above, in cylindrical polar coordinates, and derive the equivalent of (1).

16B Quantum Mechanics

If A, B, and C are operators establish the identity

$$[AB, C] = A[B, C] + [A, C]B \; .$$

A particle moves in a two-dimensional harmonic oscillator potential with Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}(x^2 + y^2) \; .$$

The angular momentum operator is defined by

$$L = xp_y - yp_x \; .$$

Show that L is hermitian and hence that its eigenvalues are real. Establish the commutation relation $[L, H] = 0$. Why does this ensure that eigenstates of H can also be chosen to be eigenstates of L?

Let $\phi_0(x, y) = e^{-(x^2+y^2)/2\hbar}$, and show that ϕ_0, $\phi_x = x\phi_0$ and $\phi_y = y\phi_0$ are all eigenstates of H, and find their respective eigenvalues. Show that

$$L\phi_0 = 0, \quad L\phi_x = i\hbar\phi_y, \quad L\phi_y = -i\hbar\phi_x \; ,$$

and hence, by taking suitable linear combinations of ϕ_x and ϕ_y, find two states, ψ_1 and ψ_2, satisfying

$$L\psi_j = \lambda_j\psi_j, \quad H\psi_j = E_j\psi_j \qquad j = 1, 2 \; .$$

Show that ψ_1 and ψ_2 are orthogonal, and find $\lambda_1, \lambda_2, E_1$ and E_2.

The particle has charge e, and an electric field of strength $\mathcal{E}$ is applied in the x-direction so that the Hamiltonian is now H', where

$$H' = H - e\mathcal{E}x \; .$$

Show that $[L, H'] = -i\hbar e\mathcal{E}y$. Why does this mean that L and H' cannot have simultaneous eigenstates?

By making the change of coordinates $x' = x - e\mathcal{E}$, $y' = y$, show that $\psi_1(x', y')$ and $\psi_2(x', y')$ are eigenstates of H' and write down the corresponding energy eigenvalues.

Find a modified angular momentum operator L' for which $\psi_1(x', y')$ and $\psi_2(x', y')$ are also eigenstates.

17A Electromagnetism

Two long thin concentric perfectly conducting cylindrical shells of radii a and b $(a < b)$ are connected together at one end by a resistor of resistance R, and at the other by a battery that establishes a potential difference V. Thus, a current $I = V/R$ flows in opposite directions along each of the cylinders.

(a) Using Ampère's law, find the magnetic field $\mathbf{B}$ in between the cylinders.

(b) Using Gauss's law and the integral relationship between the potential and the electric field, or otherwise, show that the charge per unit length on the inner cylinder is

$$\lambda = \frac{2\pi\epsilon_0 V}{\ln(b/a)},$$

and also calculate the radial electric field.

(c) Calculate the Poynting vector and by suitable integration verify that the power delivered by the system is V^2/R.

18D Fluid Dynamics

Starting from Euler's equations for an inviscid incompressible fluid of density ρ with no body force, undergoing irrotational motion, show that the pressure p is given by

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = F(t),$$

for some function $F(t)$, where ϕ is the velocity potential.

The fluid occupies an infinite domain and contains a spherical gas bubble of radius $R(t)$ in which the pressure is p_g. The pressure in the fluid at infinity is p_∞.

Show that

$$\ddot{R}\,R + \frac{3}{2}\,\dot{R}^2 = \frac{p_g - p_\infty}{\rho}\,.$$

The bubble contains a fixed mass M of gas in which

$$p_g = C\left(M/R^3\right)^2$$

for a constant C. At time $t=0, R=R_0, \dot{R}=0$ and $p_g = p_\infty/2$. Show that

$$\dot{R}^2 R^3 = \frac{p_\infty}{\rho}\left[R_0^3 - \frac{R_0^6}{3R^3} - \frac{2}{3}\,R^3\right],$$

and deduce that the bubble radius oscillates between R_0 and $R_0/2^{1/3}$.

19C Numerical Analysis

Starting from Taylor's theorem with integral form of the remainder, prove the Peano kernel theorem: the error of an approximant $L(f)$ applied to $f(x) \in C^{k+1}[a,b]$ can be written in the form

$$L(f) = \frac{1}{k!}\int_a^b K(\theta) f^{(k+1)}(\theta) d\theta.$$

You should specify the form of $K(\theta)$. Here it is assumed that $L(f)$ is identically zero when $f(x)$ is a polynomial of degree k. State any other necessary conditions.

Setting $a = 0$ and $b = 2$, find $K(\theta)$ and show that it is negative for $0 < \theta < 2$ when

$$L(f) = \int_0^2 f(x)dx - \frac{1}{3}\left(f(0) + 4f(1) + f(2)\right) \quad \text{for} \quad f(x) \in C^4[0,2].$$

Hence determine the minimum value of ρ for which

$$|L(f)| \leqslant \rho \|f^{(4)}\|_\infty,$$

holds for all $f(x) \in C^4[0,2]$.

20H Optimization

Four factories supply stuff to four shops. The production capacities of the factories are 7, 12, 8 and 9 units per week, and the requirements of the shops are 8 units per week each. If the costs of transporting a unit of stuff from factory i to shop j is the (i, j)th element in the matrix

$$\begin{pmatrix} 6 & 10 & 3 & 5 \\ 4 & 8 & 6 & 12 \\ 3 & 4 & 9 & 2 \\ 5 & 7 & 2 & 6 \end{pmatrix}$$

find a minimal-cost allocation of the outputs of the factories to the shops.

Suppose that the cost of producing one unit of stuff varies across the factories, being 3, 2, 4, 5 respectively. Explain how you would modify the original problem to minimise the total cost of production and of transportation, and find an optimal solution for the modified problem.

END OF PAPER