UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday 5 June 2008 9.00 to 12.00

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIRMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Groups, Rings and Modules

Let G be the abelian group generated by elements a, b, c, d subject to the relations

$$4a - 2b + 2c + 12d = 0, \quad -2b + 2c = 0, \quad 2b + 2c = 0, \quad 8a + 4c + 24d = 0\,.$$

Express G as a product of cyclic groups, and find the number of elements of G of order 2.

2G Geometry

A smooth surface in $\mathbb{R}^3$ has parametrization

$$\sigma(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2 v, u^2 - v^2\right).$$

Show that a unit normal vector at the point $\sigma(u,v)$ is

$$\left(\frac{-2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)$$

and that the curvature is $\dfrac{-4}{(1+u^2+v^2)^4}$.

3F Analysis II

Explain what it means for a function $f(x,y)$ of two variables to be differentiable at a point (x_0,y_0). If f is differentiable at (x_0,y_0), show that for any α the function g_α defined by

$$g_\alpha(t) = f(x_0 + t\cos\alpha, y_0 + t\sin\alpha)$$

is differentiable at $t=0$, and find its derivative in terms of the partial derivatives of f at (x_0,y_0).

Consider the function f defined by

$$\begin{aligned} f(x,y) &= (x^2y + xy^2)/(x^2+y^2) && ((x,y) \neq (0,0)) \\ &= 0 && ((x,y) = (0,0)). \end{aligned}$$

Is f differentiable at $(0,0)$? Justify your answer.

4F Metric and Topological Spaces

Explain what it means for a topological space to be connected.

Are the following subspaces of the unit square $[0,1] \times [0,1]$ connected? Justify your answers.

(*a*) $\{(x, y) : x \neq 0, y \neq 0, \text{ and } x/y \in \mathbb{Q}\}$.

(*b*) $\{(x, y) : (x = 0) \text{ or } (x \neq 0 \text{ and } y \in \mathbb{Q})\}$.

5C Complex Methods

Using the contour integration formula for the inversion of Laplace transforms find the inverse Laplace transforms of the following functions:

$$\text{(a)} \qquad \frac{s}{s^2 + a^2} \quad (a \text{ real and non–zero}), \qquad \text{(b)} \qquad \frac{1}{\sqrt{s}} \ .$$

[*You may use the fact that* $\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\pi/b}$.]

6D Methods

Let $\mathcal{L}$ be the operator

$$\mathcal{L}y = \frac{d^2y}{dx^2} - k^2 y$$

on functions $y(x)$ satisfying $\lim_{x\to-\infty} y(x) = 0$ and $\lim_{x\to\infty} y(x) = 0$.

Given that the Green's function $G(x;\xi)$ for $\mathcal{L}$ satisfies

$$\mathcal{L}G = \delta(x - \xi),$$

show that a solution of

$$\mathcal{L}y = S(x),$$

for a given function $S(x)$, is given by

$$y(x) = \int_{-\infty}^{\infty} G(x;\xi) S(\xi) d\xi.$$

Indicate why this solution is unique.

Show further that the Green's function is given by

$$G(x;\xi) = -\frac{1}{2|k|} \exp(-|k||x - \xi|).$$

7A Quantum Mechanics

Write down a formula for the orbital angular momentum operator $\hat{\mathbf{L}}$. Show that its components satisfy

$$[L_i, L_j] = i\hbar\, \epsilon_{ijk}\, L_k\,.$$

If $L_3\psi = 0$, show that $(L_1 \pm iL_2)\psi$ are also eigenvectors of L_3, and find their eigenvalues.

8H Statistics

If $X_1, \ldots, X_n$ is a sample from a density $f(\cdot|\theta)$ with θ unknown, what is a 95% confidence set for θ?

In the case where the X_i are independent $N(\mu, \sigma^2)$ random variables with σ^2 known, μ unknown, find (in terms of σ^2) how large the size n of the sample must be in order for there to exist a 95% confidence interval for μ of length no more than some given $\varepsilon > 0$.

[*Hint: If* $Z \sim N(0,1)$ *then* $P(Z > 1.960) = 0.025$.]

9H Markov Chains

What does it mean to say that a Markov chain is recurrent?

Stating clearly any general results to which you appeal, prove that the symmetric simple random walk on $\mathbb{Z}$ is recurrent.

SECTION II

10E Linear Algebra

Let $k = \mathbb{R}$ or $\mathbb{C}$. What is meant by a quadratic form $q : k^n \to k$? Show that there is a basis $\{v_1, \ldots, v_n\}$ for k^n such that, writing $x = x_1 v_1 + \ldots + x_n v_n$, we have $q(x) = a_1 x_1^2 + \ldots + a_n x_n^2$ for some scalars $a_1, \ldots, a_n \in \{-1, 0, 1\}$.

Suppose that $k = \mathbb{R}$. Define the rank and signature of q and compute these quantities for the form $q : \mathbb{R}^3 \to \mathbb{R}$ given by $q(x) = -3x_1^2 + x_2^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$.

Suppose now that $k = \mathbb{C}$ and that $q_1, \ldots, q_d : \mathbb{C}^n \to \mathbb{C}$ are quadratic forms. If $n \geqslant 2^d$, show that there is some nonzero $x \in \mathbb{C}^n$ such that $q_1(x) = \ldots = q_d(x) = 0$.

11G Groups, Rings and Modules

What is a Euclidean domain? Show that a Euclidean domain is a principal ideal domain.

Show that $\mathbb{Z}[\sqrt{-7}]$ is not a Euclidean domain (for any choice of norm), but that the ring

$$\mathbb{Z}\Big[\frac{1+\sqrt{-7}}{2}\Big]$$

is Euclidean for the norm function $N(z) = z\bar{z}$.

12G Geometry

Let D be the unit disc model of the hyperbolic plane, with metric

$$\frac{4\,|d\zeta|^2}{(1-|\zeta|^2)^2}.$$

(i) Show that the group of Möbius transformations mapping D to itself is the group of transformations

$$\zeta \mapsto \omega \frac{\zeta - \lambda}{\bar{\lambda}\zeta - 1},$$

where $|\lambda| < 1$ and $|\omega| = 1$.

(ii) Assuming that the transformations in (i) are isometries of D, show that any hyperbolic circle in D is a Euclidean circle.

(iii) Let P and Q be points on the unit circle with $\angle POQ = 2\alpha$. Show that the hyperbolic distance from O to the hyperbolic line PQ is given by

$$2\tanh^{-1}\left(\frac{1-\sin\alpha}{\cos\alpha}\right).$$

(iv) Deduce that if $a > 2\tanh^{-1}(2-\sqrt{3})$ then no hyperbolic open disc of radius a is contained in a hyperbolic triangle.

13F Analysis II

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function, and (x_0, y_0) a point of $\mathbb{R}^2$. Prove that if the partial derivatives of f exist in some open disc around (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Now let X denote the vector space of all $(n \times n)$ real matrices, and let $f\colon X \to \mathbb{R}$ be the function assigning to each matrix its determinant. Show that f is differentiable at the identity matrix I, and that $Df|_I$ is the linear map $H \mapsto \operatorname{tr} H$. Deduce that f is differentiable at any invertible matrix A, and that $Df|_A$ is the linear map $H \mapsto \det A \operatorname{tr}(A^{-1}H)$.

Show also that if K is a matrix with $\|K\| < 1$, then $(I+K)$ is invertible. Deduce that f is twice differentiable at I, and find $D^2 f|_I$ as a bilinear map $X \times X \to \mathbb{R}$.

[*You may assume that the norm* $\|-\|$ *on* X *is complete, and that it satisfies the inequality* $\|AB\| \leqslant \|A\|.\|B\|$ *for any two matrices* A *and* B.]

14E Complex Analysis

State and prove Rouché's theorem, and use it to count the number of zeros of $3z^9 + 8z^6 + z^5 + 2z^3 + 1$ inside the annulus $\{z : 1 < |z| < 2\}$.

Let $(p_n)_{n=1}^{\infty}$ be a sequence of polynomials of degree at most d with the property that $p_n(z)$ converges uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. Prove that there is a polynomial p of degree at most d such that $p_n \to p$ uniformly on compact subsets of $\mathbb{C}$. [*If you use any results about uniform convergence of analytic functions, you should prove them.*]

Suppose that p has d distinct roots $z_1, \ldots, z_d$. Using Rouché's theorem, or otherwise, show that for each i there is a sequence $(z_{i,n})_{n=1}^{\infty}$ such that $p_n(z_{i,n}) = 0$ and $z_{i,n} \to z_i$ as $n \to \infty$.

15D Methods

Let $\lambda_1 < \lambda_2 < \ldots \lambda_n \ldots$ and $y_1(x), y_2(x), \ldots y_n(x) \ldots$ be the eigenvalues and corresponding eigenfunctions for the Sturm–Liouville system

$$\mathcal{L} y_n = \lambda_n w(x) y_n ,$$

where

$$\mathcal{L} y \equiv \frac{d}{dx}\left(-p(x)\frac{dy}{dx}\right) + q(x) y ,$$

with $p(x) > 0$ and $w(x) > 0$. The boundary conditions on y are that $y(0) = y(1) = 0$.

Show that two distinct eigenfunctions are orthogonal in the sense that

$$\int_0^1 w y_n y_m \, dx = \delta_{nm} \int_0^1 w y_n^2 \, dx .$$

Show also that if y has the form

$$y = \sum_{n=1}^{\infty} a_n y_n ,$$

with a_n being independent of x, then

$$\frac{\int_0^1 y \mathcal{L} y \, dx}{\int_0^1 w y^2 \, dx} \geq \lambda_1 .$$

Assuming that the eigenfunctions are complete, deduce that a solution of the diffusion equation,

$$\frac{\partial y}{\partial t} = -\frac{1}{w} \mathcal{L} y ,$$

that satisfies the boundary conditions given above is such that

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^1 w y^2 \, dx\right) \leq -\lambda_1 \int_0^1 w y^2 \, dx .$$

16A Quantum Mechanics

What is the probability current for a particle of mass m, wavefunction ψ, moving in one dimension?

A particle of energy E is incident from $x < 0$ on a barrier given by

$$V(x) = \begin{cases} 0 & x \leqslant 0 \\ V_1 & 0 < x < a \\ V_0 & x \geqslant a \end{cases}$$

where $V_1 > V_0 > 0$. What are the conditions satisfied by ψ at $x = 0$ and $x = a$? Write down the form taken by the wavefunction in the regions $x \leqslant 0$ and $x \geqslant a$ distinguishing between the cases $E > V_0$ and $E < V_0$. For both cases, use your expressions for ψ to calculate the probability currents in these two regions.

Define the reflection and transmission coefficients, R and T. Using current conservation, show that the expressions you have derived satisfy $R + T = 1$. Show that $T = 0$ if $0 < E < V_0$.

17B Electromagnetism

(i) From Maxwell's equations in vacuum,

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} ,$$

obtain the wave equation for the electric field $\mathbf{E}$. [*You may find the following identity useful:* $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.]

(ii) If the electric and magnetic fields of a monochromatic plane wave in vacuum are

$$\mathbf{E}(z,t) = \mathbf{E}_0 \mathrm{e}^{i(kz-\omega t)} \quad \text{and} \quad \mathbf{B}(z,t) = \mathbf{B}_0 \mathrm{e}^{i(kz-\omega t)} ,$$

show that the corresponding electromagnetic waves are transverse (that is, both fields have no component in the direction of propagation).

(iii) Use Faraday's law for these fields to show that

$$\mathbf{B}_0 = \frac{k}{\omega}(\hat{\mathbf{e}}_z \times \mathbf{E}_0).$$

(iv) Explain with symmetry arguments how these results generalise to

$$\mathbf{E}(\mathbf{r},t) = E_0 \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{\mathbf{n}} \quad \text{and} \quad \mathbf{B}(\mathbf{r},t) = \frac{1}{c} E_0 \mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}} \times \hat{\mathbf{n}}) ,$$

where $\hat{\mathbf{n}}$ is the polarisation vector, *i.e.*, the unit vector perpendicular to the direction of motion and along the direction of the electric field, and $\hat{\mathbf{k}}$ is the unit vector in the direction of propagation of the wave.

(v) Using Maxwell's equations in vacuum prove that:

$$\oint_{\mathcal{A}} (1/\mu_0)(\mathbf{E} \times \mathbf{B}) \cdot d\mathcal{A} = -\frac{\partial}{\partial t} \int_{\mathcal{V}} \left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) dV \ , \qquad (1)$$

where $\mathcal{V}$ is the closed volume and $\mathcal{A}$ is the bounding surface. Comment on the differing time dependencies of the left-hand-side of (1) for the case of (a) linearly-polarized and (b) circularly-polarized monochromatic plane waves.

18B Fluid Dynamics

An ideal liquid contained within a closed circular cylinder of radius a rotates about the axis of the cylinder (assume this axis to be in the vertical z-direction).

(i) Prove that the equation of continuity and the boundary conditions are satisfied by the velocity $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$, where $\mathbf{\Omega} = \Omega \hat{\mathbf{e}}_z$ is the angular velocity, with $\hat{\mathbf{e}}_z$ the unit vector in the z-direction, which depends only on time, and $\mathbf{r}$ is the position vector measured from a point on the axis of rotation.

(ii) Calculate the angular momentum $\mathbf{M} = \rho \int (\mathbf{r} \times \mathbf{v}) dV$ per unit length of the cylinder.

(iii) Suppose the the liquid starts from rest and flows under the action of an external force per unit mass $\mathbf{f} = (\alpha x + \beta y, \gamma x + \delta y, 0)$. By taking the curl of the Euler equation, prove that

$$\frac{d\Omega}{dt} = \frac{1}{2}(\gamma - \beta) \ .$$

(iv) Find the pressure.

19D Numerical Analysis

Starting from the Taylor formula for $f(x) \in C^{k+1}[a, b]$ with an integral remainder term, show that the error of an approximant $L(f)$ can be written in the form (Peano kernel theorem)

$$L(f) = \frac{1}{k!} \int_a^b K(\theta) f^{(k+1)}(\theta) d\theta,$$

when $L(f)$, which is identically zero if $f(x)$ is a polynomial of degree k, satisfies conditions that you should specify. Give an expression for $K(\theta)$.

Hence determine the minimum value of c in the inequality

$$|L(f)| \leq c \|f'''\|_\infty \ ,$$

when

$$L(f) = f'(1) - \frac{1}{2}\left(f(2) - f(0)\right) \quad \text{for} \quad f(x) \in C^3[0, 2].$$

20H Optimization

Use the simplex algorithm to solve the problem

$$\max \ x_1 + 2x_2 - 6x_3$$

subject to $x_1, x_2 \geqslant 0$, $|x_3| \leqslant 5$, and

$$\begin{aligned} x_1 + x_2 + x_3 &\leqslant 7 \, , \\ 2x_2 + x_3 &\geqslant 1 \, . \end{aligned}$$

END OF PAPER