UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2007

List of Courses

1/I/1G **Linear Algebra**

Suppose that $\{e_1, \ldots, e_3\}$ is a basis of the complex vector space $\mathbb{C}^3$ and that $A : \mathbb{C}^3 \to \mathbb{C}^3$ is the linear operator defined by $A(e_1) = e_2$, $A(e_2) = e_3$, and $A(e_3) = e_1$.

By considering the action of A on column vectors of the form $(1, \xi, \xi^2)^T$, where $\xi^3 = 1$, or otherwise, find the diagonalization of A and its characteristic polynomial.

1/II/9G **Linear Algebra**

State and prove Sylvester's law of inertia for a real quadratic form.

[*You may assume that for each real symmetric matrix A there is an orthogonal matrix U, such that $U^{-1}AU$ is diagonal.*]

Suppose that V is a real vector space of even dimension $2m$, that Q is a non-singular quadratic form on V and that U is an m-dimensional subspace of V on which Q vanishes. What is the signature of Q?

2/I/1G **Linear Algebra**

Suppose that S, T are endomorphisms of the 3-dimensional complex vector space $\mathbb{C}^3$ and that the eigenvalues of each of them are 1, 2, 3. What are their characteristic and minimal polynomials? Are they conjugate?

2/II/10G **Linear Algebra**

Suppose that P is the complex vector space of complex polynomials in one variable, z.

(i) Show that the form $\langle\ ,\ \rangle$ defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{i\theta}\right) \cdot \overline{g\left(e^{i\theta}\right)}\, d\theta$$

is a positive definite Hermitian form on P.

(ii) Find an orthonormal basis of P for this form, in terms of the powers of z.

(iii) Generalize this construction to complex vector spaces of complex polynomials in any finite number of variables.

3/II/10G **Linear Algebra**

(i) Define the terms *row-rank*, *column-rank* and *rank* of a matrix, and state a relation between them.

(ii) Fix positive integers m, n, p with $m, n \geqslant p$. Suppose that A is an $m \times p$ matrix and B a $p \times n$ matrix. State and prove the best possible upper bound on the rank of the product AB.

4/I/1G **Linear Algebra**

Suppose that $\alpha : V \to W$ is a linear map of finite-dimensional complex vector spaces. What is the dual map α^* of the dual vector spaces?

Suppose that we choose bases of V, W and take the corresponding dual bases of the dual vector spaces. What is the relation between the matrices that represent α and α^* with respect to these bases? Justify your answer.

4/II/10G **Linear Algebra**

(i) State and prove the Cayley–Hamilton theorem for square complex matrices.

(ii) A square matrix A is *of order* n for a strictly positive integer n if $A^n = I$ and no smaller positive power of A is equal to I.

Determine the order of a complex 2×2 matrix A of trace zero and determinant 1.

1/II/10G Groups, Rings and Modules

(i) State a structure theorem for finitely generated abelian groups.

(ii) If K is a field and f a polynomial of degree n in one variable over K, what is the maximal number of zeroes of f? Justify your answer in terms of unique factorization in some polynomial ring, or otherwise.

(iii) Show that any finite subgroup of the multiplicative group of non-zero elements of a field is cyclic. Is this true if the subgroup is allowed to be infinite?

2/I/2G Groups, Rings and Modules

Define the term *Euclidean domain*.

Show that the ring of integers $\mathbb{Z}$ is a Euclidean domain.

2/II/11G Groups, Rings and Modules

(i) Give an example of a Noetherian ring and of a ring that is not Noetherian. Justify your answers.

(ii) State and prove Hilbert's basis theorem.

3/I/1G Groups, Rings and Modules

What are the orders of the groups $GL_2(\mathbb{F}_p)$ and $SL_2(\mathbb{F}_p)$ where $\mathbb{F}_p$ is the field of p elements?

3/II/11G Groups, Rings and Modules

(i) State the Sylow theorems for Sylow p-subgroups of a finite group.

(ii) Write down one Sylow 3-subgroup of the symmetric group S_5 on 5 letters. Calculate the number of Sylow 3-subgroups of S_5.

4/I/2G Groups, Rings and Modules

If p is a prime, how many abelian groups of order p^4 are there, up to isomorphism?

4/II/11G **Groups, Rings and Modules**

A regular icosahedron has 20 faces, 12 vertices and 30 edges. The group G of its rotations acts transitively on the set of faces, on the set of vertices and on the set of edges.

(i) List the conjugacy classes in G and give the size of each.

(ii) Find the order of G and list its normal subgroups.

[*A normal subgroup of* G *is a union of conjugacy classes in* G.]

1/I/2A **Geometry**

State the Gauss–Bonnet theorem for spherical triangles, and deduce from it that for each convex polyhedron with F faces, E edges, and V vertices, $F - E + V = 2$.

2/II/12A **Geometry**

(i) The spherical circle with centre $P \in S^2$ and radius r, $0 < r < \pi$, is the set of all points on the unit sphere S^2 at spherical distance r from P. Find the circumference of a spherical circle with spherical radius r. Compare, for small r, with the formula for a Euclidean circle and comment on the result.

(ii) The cross ratio of four distinct points z_i in $\mathbf{C}$ is

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_4 - z_3)(z_2 - z_1)} \,.$$

Show that the cross-ratio is a real number if and only if z_1, z_2, z_3, z_4 lie on a circle or a line.

[*You may assume that Möbius transformations preserve the cross-ratio.*]

3/I/2A **Geometry**

Let l be a line in the Euclidean plane $\mathbf{R}^2$ and P a point on l. Denote by ρ the reflection in l and by τ the rotation through an angle α about P. Describe, in terms of l, P, and α, a line fixed by the composition $\tau\rho$ and show that $\tau\rho$ is a reflection.

3/II/12A **Geometry**

For a parameterized smooth embedded surface $\sigma : V \to U \subset \mathbf{R}^3$, where V is an open domain in $\mathbf{R}^2$, define the *first fundamental form*, the *second fundamental form*, and the *Gaussian curvature* K. State the Gauss–Bonnet formula for a compact embedded surface $S \subset \mathbf{R}^3$ having Euler number $e(S)$.

Let S denote a surface defined by rotating a curve

$$\eta(u) = (r + a \sin u, 0, b \cos u) \qquad 0 \leq u \leq 2\pi \,,$$

about the z-axis. Here a, b, r are positive constants, such that $a^2 + b^2 = 1$ and $a < r$. By considering a smooth parameterization, find the first fundamental form and the second fundamental form of S.

4/II/12A **Geometry**

Write down the Riemannian metric for the upper half-plane model **H** of the hyperbolic plane. Describe, without proof, the group of isometries of **H** and the hyperbolic lines (i.e. the geodesics) on **H**.

Show that for any two hyperbolic lines ℓ_1, ℓ_2, there is an isometry of **H** which maps ℓ_1 onto ℓ_2.

Suppose that g is a composition of two reflections in hyperbolic lines which are ultraparallel (i.e. do not meet either in the hyperbolic plane or at its boundary). Show that g cannot be an element of finite order in the group of isometries of **H**.

[*Existence of a common perpendicular to two ultraparallel hyperbolic lines may be assumed. You might like to choose carefully which hyperbolic line to consider as a common perpendicular.*]

1/II/11H **Analysis II**

Define what it means for a function $f : \mathbb{R}^a \to \mathbb{R}^b$ to be differentiable at a point $p \in \mathbb{R}^a$ with derivative a linear map $Df|_p$.

State the Chain Rule for differentiable maps $f : \mathbb{R}^a \to \mathbb{R}^b$ and $g : \mathbb{R}^b \to \mathbb{R}^c$. Prove the Chain Rule.

Let $\|x\|$ denote the standard Euclidean norm of $x \in \mathbb{R}^a$. Find the partial derivatives $\frac{\partial f}{\partial x_i}$ of the function $f(x) = \|x\|$ where they exist.

2/I/3H **Analysis II**

For integers a and b, define $d(a, b)$ to be 0 if $a = b$, or $\frac{1}{2^n}$ if $a \neq b$ and n is the largest non-negative integer such that $a - b$ is a multiple of 2^n. Show that d is a metric on the integers $\mathbb{Z}$.

Does the sequence $x_n = 2^n - 1$ converge in this metric?

2/II/13H **Analysis II**

Show that the limit of a uniformly convergent sequence of real valued continuous functions on $[0, 1]$ is continuous on $[0, 1]$.

Let f_n be a sequence of continuous functions on $[0, 1]$ which converge point-wise to a continuous function. Suppose also that the integrals $\int_0^1 f_n(x)\, dx$ converge to $\int_0^1 f(x)\, dx$. Must the functions f_n converge uniformly to f? Prove or give a counterexample.

Let f_n be a sequence of continuous functions on $[0, 1]$ which converge point-wise to a function f. Suppose that f is integrable and that the integrals $\int_0^1 f_n(x)\, dx$ converge to $\int_0^1 f(x)\, dx$. Is the limit f necessarily continuous? Prove or give a counterexample.

3/I/3H **Analysis II**

Define uniform continuity for a real-valued function on an interval in the real line. Is a uniformly continuous function on the real line necessarily bounded?

Which of the following functions are uniformly continuous on the real line?

(i) $f(x) = x \sin x$,

(ii) $f(x) = e^{-x^4}$.

Justify your answers.

3/II/13H Analysis II

Let V be the real vector space of continuous functions $f : [0,1] \to \mathbb{R}$. Show that defining

$$||f|| = \int_0^1 |f(x)|dx$$

makes V a normed vector space.

Define $f_n(x) = \sin nx$ for positive integers n. Is the sequence (f_n) convergent to some element of V? Is (f_n) a Cauchy sequence in V? Justify your answers.

4/I/3H Analysis II

Define uniform convergence for a sequence $f_1, f_2, \ldots$ of real-valued functions on the interval $(0,1)$.

For each of the following sequences of functions on $(0,1)$, find the pointwise limit function. Which of these sequences converge uniformly on $(0,1)$?

(i) $f_n(x) = \log\,(x + \frac{1}{n})$,

(ii) $f_n(x) = \cos\,(\frac{x}{n})$.

Justify your answers.

4/II/13H Analysis II

State and prove the Contraction Mapping Theorem.

Find numbers a and b, with $a < 0 < b$, such that the mapping $T : C[a,b] \to C[a,b]$ defined by

$$T(f)(x) = 1 + \int_0^x 3t\,f(t)\,dt$$

is a contraction, in the sup norm on $C[a,b]$. Deduce that the differential equation

$$\frac{dy}{dx} = 3xy, \qquad \text{with } y = 1 \text{ when } x = 0,$$

has a unique solution in some interval containing 0.

1/II/12A Metric and Topological Spaces

Let X and Y be topological spaces. Define the product topology on $X \times Y$ and show that if X and Y are Hausdorff then so is $X \times Y$.

Show that the following statements are equivalent.

(i) X is a Hausdorff space.

(ii) The diagonal $\Delta = \{(x, x) : x \in X\}$ is a closed subset of $X \times X$, in the product topology.

(iii) For any topological space Y and any continuous maps $f, g : Y \to X$, the set $\{y \in Y : f(y) = g(y)\}$ is closed in Y.

2/I/4A Metric and Topological Spaces

Are the following statements true or false? Give a proof or a counterexample as appropriate.

(i) If $f : X \to Y$ is a continuous map of topological spaces and $S \subseteq X$ is compact then $f(S)$ is compact.

(ii) If $f : X \to Y$ is a continuous map of topological spaces and $K \subseteq Y$ is compact then $f^{-1}(K) = \{x \in X : f(x) \in K\}$ is compact.

(iii) If a metric space M is complete and a metric space T is homeomorphic to M then T is complete.

3/I/4A Metric and Topological Spaces

(a) Let X be a connected topological space such that each point x of X has a neighbourhood homeomorphic to $\mathbb{R}^n$. Prove that X is path-connected.

(b) Let τ denote the topology on $\mathbb{N} = \{1, 2, \ldots\}$, such that the open sets are $\mathbb{N}$, the empty set, and all the sets $\{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$. Prove that any continuous map from the topological space $(\mathbb{N}, \tau)$ to the Euclidean $\mathbb{R}$ is constant.

4/II/14A Metric and Topological Spaces

(a) For a subset A of a topological space X, define the *closure* $cl(A)$ of A. Let $f : X \to Y$ be a map to a topological space Y. Prove that f is continuous if and only if $f(cl(A)) \subseteq cl(f(A))$, for each $A \subseteq X$.

(b) Let M be a metric space. A subset S of M is called *dense* in M if the closure of S is equal to M.

Prove that if a metric space M is compact then it has a countable subset which is dense in M.

1/I/3F **Complex Analysis or Complex Methods**

For the function

$$f(z) = \frac{2z}{z^2+1},$$

determine the Taylor series of f around the point $z_0 = 1$, and give the largest r for which this series converges in the disc $|z-1| < r$.

1/II/13F **Complex Analysis or Complex Methods**

By integrating round the contour C_R, which is the boundary of the domain

$$D_R = \{z = re^{i\theta} : 0 < r < R, \quad 0 < \theta < \frac{\pi}{4}\},$$

evaluate each of the integrals

$$\int_0^\infty \sin x^2 dx, \quad \int_0^\infty \cos x^2 dx\,.$$

[*You may use the relations* $\int_0^\infty e^{-r^2} dr = \frac{\sqrt{\pi}}{2}$ *and* $\sin t \geq \frac{2}{\pi} t$ *for* $0 \leq t \leq \frac{\pi}{2}$.]

2/II/14F **Complex Analysis or Complex Methods**

Let Ω be the half-strip in the complex plane,

$$\Omega = \{z = x + iy \in \mathbb{C} \;:\; -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y > 0\}\,.$$

Find a conformal mapping that maps Ω onto the unit disc.

3/II/14H Complex Analysis

Say that a function on the complex plane $\mathbb{C}$ is *periodic* if $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all z. If f is a periodic analytic function, show that f is constant.

If f is a meromorphic periodic function, show that the number of zeros of f in the square $[0,1) \times [0,1)$ is equal to the number of poles, both counted with multiplicities.

Define

$$f(z) = \frac{1}{z^2} + \sum_w \left[\frac{1}{(z-w)^2} - \frac{1}{w^2} \right],$$

where the sum runs over all $w = a + bi$ with a and b integers, not both 0. Show that this series converges to a meromorphic periodic function on the complex plane.

4/I/4H Complex Analysis

State the argument principle.

Show that if f is an analytic function on an open set $U \subset \mathbb{C}$ which is one-to-one, then $f'(z) \neq 0$ for all $z \in U$.

3/I/5F **Complex Methods**

Show that the function $\phi(x,y) = \tan^{-1}\dfrac{y}{x}$ is harmonic. Find its harmonic conjugate $\psi(x,y)$ and the analytic function $f(z)$ whose real part is $\phi(x,y)$. Sketch the curves $\phi(x,y) = C$ and $\psi(x,y) = K$.

4/II/15F **Complex Methods**

(i) Use the definition of the Laplace transform of $f(t)$:

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt\,,$$

to show that, for $f(t) = t^n$,

$$L\{f(t)\} = F(s) = \frac{n!}{s^{n+1}}\,, \qquad L\{e^{at} f(t)\} = F(s-a) = \frac{n!}{(s-a)^{n+1}}\,.$$

(ii) Use contour integration to find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s+1)^2}\,.$$

(iii) Verify the result in (ii) by using the results in (i) and the convolution theorem.

(iv) Use Laplace transforms to solve the differential equation

$$f^{(iv)}(t) + 2f'''(t) + f''(t) = 0,$$

subject to the initial conditions

$$f(0) = f'(0) = f''(0) = 0, \qquad f'''(0) = 1\,.$$

1/II/14D **Methods**

Define the Fourier transform $\tilde{f}(k)$ of a function $f(x)$ that tends to zero as $|x| \to \infty$, and state the inversion theorem. State and prove the convolution theorem.

Calculate the Fourier transforms of

$$\begin{array}{lll} & (i) \quad f(x) & = e^{-a|x|}, \\ \text{and} & (ii) \quad g(x) & = \begin{cases} 1\,, & |x| \leqslant b \\ 0\,, & |x| > b\,. \end{cases} \end{array}$$

Hence show that

$$\int_{-\infty}^{\infty} \frac{\sin(bk)\, e^{ikx}}{k\,(a^2+k^2)}\, dk \;=\; \frac{\pi \sinh(ab)}{a^2}\, e^{-ax} \qquad \text{for} \quad x > b\,,$$

and evaluate this integral for all other (real) values of x.

2/I/5D **Methods**

Show that a smooth function $y(x)$ that satisfies $y(0) = y'(1) = 0$ can be written as a Fourier series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n \sin \lambda_n x, \quad 0 \leqslant x \leqslant 1\,,$$

where the λ_n should be specified. Write down an integral expression for a_n.

Hence solve the following differential equation

$$y'' - \alpha^2 y = x \cos \pi x\,,$$

with boundary conditions $y(0) = y'(1) = 0$, in the form of an infinite series.

2/II/15D **Methods**

Let $y_0(x)$ be a non-zero solution of the Sturm-Liouville equation

$$L(y_0;\lambda_0) \equiv \frac{d}{dx}\left(p(x)\frac{dy_0}{dx}\right) + (q(x) + \lambda_0 w(x))\, y_0 = 0$$

with boundary conditions $y_0(0) = y_0(1) = 0$. Show that, if $y(x)$ and $f(x)$ are related by

$$L(y;\lambda_0) = f\,,$$

with $y(x)$ satisfying the same boundary conditions as $y_0(x)$, then

$$\int_0^1 y_0 f\, dx = 0\,. \qquad (*)$$

Suppose that y_0 is normalised so that

$$\int_0^1 w y_0^2 dx = 1\,,$$

and consider the problem

$$L(y;\lambda) = y^3\,; \quad y(0) = y(1) = 0\,.$$

By choosing f appropriately in $(*)$ deduce that, if

$$\lambda - \lambda_0 = \epsilon^2\mu \;\; [\mu = O(1), \epsilon \ll 1\,], \quad \text{and} \quad y(x) = \epsilon y_0(x) + \epsilon^2 y_1(x)$$

then

$$\mu = \int_0^1 y_0^4 dx + O(\epsilon)\,.$$

3/I/6E **Methods**

Describe the method of Lagrange multipliers for finding extrema of a function $f(x,y,z)$ subject to the constraint that $g(x,y,z) = c$.

Illustrate the method by finding the maximum and minimum values of xy for points (x,y,z) lying on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,,$$

with a, b and c all positive.

3/II/15E **Methods**

Legendre's equation may be written

$$\left(1-x^2\right)y''-2xy'+n(n+1)y=0 \quad \text{with} \quad y(1)=1\,.$$

Show that if n is a positive integer, this equation has a solution $y=P_n(x)$ that is a polynomial of degree n. Find P_0, P_1 and P_2 explicitly.

Write down a general separable solution of Laplace's equation, $\nabla^2\phi=0$, in spherical polar coordinates (r,θ). (A derivation of this result is *not* required.)

Hence or otherwise find ϕ when

$$\nabla^2\phi=0, \quad a<r<b\,,$$

with $\phi=\sin^2\theta$ both when $r=a$ and when $r=b$.

4/I/5B **Methods**

Show that the general solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}\,,$$

where c is a constant, is

$$y=f(x+ct)+g(x-ct)\,,$$

where f and g are twice differentiable functions. Briefly discuss the physical interpretation of this solution.

Calculate $y(x,t)$ subject to the initial conditions

$$y(x,0)=0 \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0)=\psi(x)\,.$$

4/II/16E **Methods**

Write down the Euler-Lagrange equation for extrema of the functional

$$I = \int_a^b F\left(y, y'\right) dx\,.$$

Show that a first integral of this equation is given by

$$F - y' \frac{\partial F}{\partial y'} = C\,.$$

A road is built between two points A and B in the plane $z = 0$ whose polar coordinates are $r = a$, $\theta = 0$ and $r = a$, $\theta = \pi/2$ respectively. Owing to congestion, the traffic speed at points along the road is kr^2 with k a positive constant. If the equation describing the road is $r = r(\theta)$, obtain an integral expression for the total travel time T from A to B.

[*Arc length in polar coordinates is given by* $ds^2 = dr^2 + r^2 d\theta^2$.]

Calculate T for the circular road $r = a$.

Find the equation for the road that minimises T and determine this minimum value.

1/II/15B **Quantum Mechanics**

The relative motion of a neutron and proton is described by the Schrödinger equation for a single particle of mass m under the influence of the central potential

$$V(r) = \begin{cases} -U & r < a \\ 0 & r > a, \end{cases}$$

where U and a are positive constants. Solve this equation for a spherically symmetric state of the deuteron, which is a bound state of a proton and neutron, giving the condition on U for this state to exist.

[*If* ψ *is spherically symmetric then* $\nabla^2\psi = \frac{1}{r}\,\frac{d^2}{dr^2}\,(r\psi)$.]

2/II/16B **Quantum Mechanics**

Write down the angular momentum operators L_1, L_2, L_3 in terms of the position and momentum operators, $\mathbf{x}$ and $\mathbf{p}$, and the commutation relations satisfied by $\mathbf{x}$ and $\mathbf{p}$.

Verify the commutation relations

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k\,.$$

Further, show that

$$[L_i, p_j] = i\hbar\epsilon_{ijk}p_k\,.$$

A wave-function $\Psi_0(r)$ is spherically symmetric. Verify that

$$\mathbf{L}\Psi_0(r) = 0\,.$$

Consider the vector function $\mathbf{\Phi} = \nabla\Psi_0(r)$. Show that Φ_3 and $\Phi_1 \pm i\Phi_2$ are eigenfunctions of L_3 with eigenvalues $0, \pm\hbar$ respectively.

3/I/7B Quantum Mechanics

The quantum mechanical harmonic oscillator has Hamiltonian

$$H = \frac{1}{2m}\, p^2 + \frac{1}{2}\, m\omega^2 x^2 \,,$$

and is in a stationary state of energy $< H >= E$. Show that

$$E \geqslant \frac{1}{2m}\, (\Delta p)^2 + \frac{1}{2}\, m\omega^2 (\Delta x)^2 \,,$$

where $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$ and $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$. Use the Heisenberg Uncertainty Principle to show that

$$E \geqslant \frac{1}{2}\hbar\omega \,.$$

3/II/16B Quantum Mechanics

A quantum system has a complete set of orthonormal eigenstates, $\psi_n(x)$, with non-degenerate energy eigenvalues, E_n, where $n = 1, 2, 3 \ldots$. Write down the wave-function, $\Psi(x,t), t \geqslant 0$ in terms of the eigenstates.

A linear operator acts on the system such that

$$\begin{aligned} A\psi_1 &= 2\psi_1 - \psi_2 \\ A\psi_2 &= 2\psi_2 - \psi_1 \ . \\ A\psi_n &= 0,\ n \geqslant 3 \end{aligned}$$

Find the eigenvalues of A and obtain a complete set of normalised eigenfunctions, ϕ_n, of A in terms of the ψ_n.

At time $t = 0$ a measurement is made and it is found that the observable corresponding to A has value 3. After time t, A is measured again. What is the probability that the value is found to be 1?

4/I/6B Quantum Mechanics

A particle moving in one space dimension with wave-function $\Psi(x,t)$ obeys the time-dependent Schrödinger equation. Write down the probability density, ρ, and current density, j, in terms of the wave-function and show that they obey the equation

$$\frac{\partial j}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \,.$$

The wave-function is

$$\Psi(x,t) = \left(e^{ikx} + R\, e^{-ikx}\right) e^{-iEt/\hbar} \,,$$

where $E = \hbar^2 k^2 / 2m$ and R is a constant, which may be complex. Evaluate j.

1/II/16E **Electromagnetism**

A steady magnetic field $\mathbf{B}(\mathbf{x})$ is generated by a current distribution $\mathbf{j}(\mathbf{x})$ that vanishes outside a bounded region V. Use the divergence theorem to show that

$$\int_V \mathbf{j}\, dV = 0 \quad \text{and} \quad \int_V x_i j_k\, dV = -\int_V x_k j_i\, dV.$$

Define the *magnetic vector potential* $\mathbf{A}(\mathbf{x})$. Use Maxwell's equations to obtain a differential equation for $\mathbf{A}(\mathbf{x})$ in terms of $\mathbf{j}(\mathbf{x})$.

It may be shown that for an unbounded domain the equation for $\mathbf{A}(\mathbf{x})$ has solution

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{j}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, dV'\,.$$

Deduce that in general the leading order approximation for $\mathbf{A}(\mathbf{x})$ as $|\mathbf{x}| \to \infty$ is

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3} \quad \text{where} \quad \mathbf{m} = \frac{1}{2} \int_V \mathbf{x}' \times \mathbf{j}(\mathbf{x}')\, dV'\,.$$

Find the corresponding far-field expression for $\mathbf{B}(\mathbf{x})$.

2/I/6E **Electromagnetism**

A metal has uniform conductivity σ. A cylindrical wire with radius a and length l is manufactured from the metal. Show, using Maxwell's equations, that when a steady current I flows along the wire the current density within the wire is uniform.

Deduce the electrical resistance of the wire and the rate of Ohmic dissipation within it.

Indicate briefly, and without detailed calculation, whether your results would be affected if the wire was not straight.

2/II/17E **Electromagnetism**

If S is a fixed surface enclosing a volume V, use Maxwell's equations to show that

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right) dV + \int_S \mathbf{P}\cdot\mathbf{n}\, dS = -\int_V \mathbf{j}\cdot\mathbf{E}\, dV\,,$$

where $\mathbf{P} = (\mathbf{E}\times\mathbf{B})/\mu_0$. Give a physical interpretation of each term in this equation.

Show that Maxwell's equations for a vacuum permit plane wave solutions with $\mathbf{E} = E_0(0,1,0)\cos(kx-\omega t)$ with E_0, k and ω constants, and determine the relationship between k and ω.

Find also the corresponding $\mathbf{B}(\mathbf{x},t)$ and hence the time average $<\mathbf{P}>$. What does $<\mathbf{P}>$ represent in this case?

3/II/17E **Electromagnetism**

A capacitor consists of three long concentric cylinders of radii a, λa and $2a$ respectively, where $1<\lambda<2$. The inner and outer cylinders are earthed (i.e. held at zero potential); the cylinder of radius λa is raised to a potential V. Find the electrostatic potential in the regions between the cylinders and deduce the capacitance, $C(\lambda)$ per unit length, of the system.

For $\lambda = 1+\delta$ with $0<\delta\ll 1$ find $C(\lambda)$ correct to leading order in δ and comment on your result.

Find also the value of λ at which $C(\lambda)$ has an extremum. Is the extremum a maximum or a minimum? Justify your answer.

4/I/7E **Electromagnetism**

Write down Faraday's law of electromagnetic induction for a moving circuit $C(t)$ in a magnetic field $\mathbf{B}(\mathbf{x},t)$. Explain carefully the meaning of each term in the equation.

A thin wire is bent into a circular loop of radius a. The loop lies in the (x,z)-plane at time $t=0$. It spins steadily with angular velocity $\Omega\mathbf{k}$, where Ω is a constant and $\mathbf{k}$ is a unit vector in the z-direction. A spatially uniform magnetic field $\mathbf{B} = B_0(\cos\omega t, \sin\omega t, 0)$ is applied, with B_0 and ω both constant. If the resistance of the wire is R, find the current in the wire at time t.

1/I/4B **Special Relativity**

Write down the position four-vector. Suppose this represents the position of a particle with rest mass M and velocity $\mathbf{v}$. Show that the four momentum of the particle is

$$p_a = (M\gamma c,\ M\gamma \mathbf{v}),$$

where $\gamma = \left(1 - |\mathbf{v}|^2/c^2\right)^{-1/2}$.

For a particle of zero rest mass show that

$$p_a = (|\mathbf{p}|,\ \mathbf{p}),$$

where $\mathbf{p}$ is the three momentum.

2/I/7B **Special Relativity**

A particle in inertial frame S has coordinates (t, x), whilst the coordinates are (t', x') in frame S', which moves with relative velocity v in the x direction. What is the relationship between the coordinates of S and S'?

Frame S'', with cooordinates (t'', x''), moves with velocity u with respect to S' and velocity V with respect to S. Derive the relativistic formula for V in terms of u and v. Show how the Newtonian limit is recovered.

4/II/17B **Special Relativity**

(a) A moving π^0 particle of rest-mass m_π decays into two photons of zero rest-mass,

$$\pi^0 \to \gamma + \gamma.$$

Show that

$$\sin\frac{\theta}{2} = \frac{m_\pi c^2}{2\sqrt{E_1 E_2}},$$

where θ is the angle between the three-momenta of the two photons and E_1, E_2 are their energies.

(b) The π^- particle of rest-mass m_π decays into an electron of rest-mass m_e and a neutrino of zero rest mass,

$$\pi^- \to e^- + \nu.$$

Show that v, the speed of the electron in the rest frame of the π^-, is

$$v = c\left[\frac{1-(m_e/m_\pi)^2}{1+(m_e/m_\pi)^2}\right].$$

1/I/5D **Fluid Dynamics**

A steady two-dimensional velocity field is given by

$$\mathbf{u}(x, y) = (\alpha x - \beta y\,,\, \beta x - \alpha y)\,, \quad \alpha > 0\,, \quad \beta > 0\,.$$

(i) Calculate the vorticity of the flow.

(ii) Verify that $\mathbf{u}$ is a possible flow field for an incompressible fluid, and calculate the stream function.

(iii) Show that the streamlines are bounded if and only if $\alpha < \beta$.

(iv) What are the streamlines in the case $\alpha = \beta$?

1/II/17D **Fluid Dynamics**

Write down the Euler equation for the steady motion of an inviscid, incompressible fluid in a constant gravitational field. From this equation, derive (a) Bernoulli's equation and (b) the integral form of the momentum equation for a fixed control volume V with surface S.

(i) A circular jet of water is projected vertically upwards with speed U_0 from a nozzle of cross-sectional area A_0 at height $z = 0$. Calculate how the speed U and cross-sectional area A of the jet vary with z, for $z \ll U_0^2/2g$.

(ii) A circular jet of speed U and cross-sectional area A impinges axisymmetrically on the vertex of a cone of semi-angle α, spreading out to form an almost parallel-sided sheet on the surface. Choose a suitable control volume and, neglecting gravity, show that the force exerted by the jet on the cone is

$$\rho A U^2 (1 - \cos\alpha)\,.$$

(iii) A cone of mass M is supported, axisymmetrically and vertex down, by the jet of part (i), with its vertex at height $z = h$, where $h \ll U_0^2/2g$. Assuming that the result of part (ii) still holds, show that h is given by

$$\rho A_0 U_0^2 \left(1 - \frac{2gh}{U_0^2}\right)^{\frac{1}{2}} (1 - \cos\alpha) \;=\; Mg\,.$$

2/I/8D **Fluid Dynamics**

An incompressible, inviscid fluid occupies the region beneath the free surface $y = \eta(x,t)$ and moves with a velocity field given by the velocity potential $\phi(x,y,t)$; gravity acts in the $-y$ direction. Derive the kinematic and dynamic boundary conditions that must be satisfied by ϕ on $y = \eta(x,t)$.

[*You may assume Bernoulli's integral of the equation of motion:*

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}\,|\nabla\phi|^2 + gy \;=\; F(t)\,.\,]$$

In the absence of waves, the fluid has uniform velocity U in the x direction. Derive the linearised form of the above boundary conditions for small amplitude waves, and verify that they and Laplace's equation are satisfied by the velocity potential

$$\phi \;=\; Ux + Re\{be^{ky}e^{i(kx-\omega t)}\}\,,$$

where $|kb| \ll U$, with a corresponding expression for η, as long as

$$(\omega - kU)^2 \;=\; gk\,.$$

What are the propagation speeds of waves with a given wave-number k?

3/II/18D **Fluid Dynamics**

Given that the circulation round every closed material curve in an inviscid, incompressible fluid remains constant in time, show that the velocity field of such a fluid started from rest can be written as the gradient of a potential, ϕ, that satisfies Laplace's equation.

A rigid sphere of radius a moves in a straight line at speed U in a fluid that is at rest at infinity. Using axisymmetric spherical polar coordinates (r, θ), with $\theta = 0$ in the direction of motion, write down the boundary conditions on ϕ and, by looking for a solution of the form $\phi = f(r) \cos\theta$, show that the velocity potential is given by

$$\phi = \frac{-Ua^3 \cos\theta}{2r^2}.$$

Calculate the kinetic energy of the fluid.

A rigid sphere of radius a and uniform density ρ_b is submerged in an infinite fluid of density ρ, under the action of gravity. Show that, when the sphere is released from rest, its initial upwards acceleration is

$$\frac{2(\rho - \rho_b)g}{\rho + 2\rho_b}.$$

[*Laplace's equation for an axisymmetric scalar field in spherical polars is:*

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0.]$$

4/II/18D **Fluid Dynamics**

Starting from Euler's equation for an inviscid, incompressible fluid in the absence of body forces,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p,$$

derive the equation for the vorticity $\boldsymbol{\omega} = \nabla_\wedge \mathbf{u}$.

[*You may assume that* $\nabla_\wedge(\mathbf{a}_\wedge \mathbf{b}) = \mathbf{a}\nabla.\mathbf{b} - \mathbf{b}\nabla.\mathbf{a} + (\mathbf{b}.\nabla)\mathbf{a} - (\mathbf{a}.\nabla)\mathbf{b}$.]

Show that, in a two-dimensional flow, vortex lines keep their strength and move with the fluid.

Show that a two-dimensional flow driven by a line vortex of circulation Γ at distance b from a rigid plane wall is the same as if the wall were replaced by another vortex of circulation $-\Gamma$ at the image point, distance b from the wall on the other side. Deduce that the first vortex will move at speed $\Gamma/4\pi b$ parallel to the wall.

A line vortex of circulation Γ moves in a quarter-plane, bounded by rigid plane walls at $x = 0,\ y > 0$ and $y = 0,\ x > 0$. Show that the vortex follows a trajectory whose equation in plane polar coordinates is $r \sin 2\theta = \text{constant}$.

1/I/6F Numerical Analysis

Solve the least squares problem

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 4 \\ -1 \end{bmatrix}$$

using QR method with Householder transformation. (A solution using normal equations is *not* acceptable.)

2/II/18F Numerical Analysis

For a symmetric, positive definite matrix A with the spectral radius $\rho(A)$, the linear system $Ax = b$ is solved by the iterative procedure

$$x^{(k+1)} = x^{(k)} - \tau(Ax^{(k)} - b), \qquad k \geq 0\,,$$

where τ is a real parameter. Find the range of τ that guarantees convergence of $x^{(k)}$ to the exact solution for any choice of $x^{(0)}$.

3/II/19F Numerical Analysis

Prove that the monic polynomials Q_n, $n \geq 0$, orthogonal with respect to a given weight function $w(x) > 0$ on $[a, b]$, satisfy the three-term recurrence relation

$$Q_{n+1}(x) = (x - a_n)Q_n(x) - b_n Q_{n-1}(x), \quad n \geq 0\,.$$

where $Q_{-1}(x) \equiv 0$, $Q_0(x) \equiv 1$. Express the values a_n and b_n in terms of Q_n and Q_{n-1} and show that $b_n > 0$.

4/I/8F Numerical Analysis

Given $f \in C^3[0, 2]$, we approximate $f'(0)$ by the linear combination

$$\mu(f) = -\frac{3}{2}f(0) + 2f(1) - \frac{1}{2}f(2)\,.$$

Using the Peano kernel theorem, determine the least constant c in the inequality

$$|f'(0) - \mu(f)| \leq c\,\|f'''\|_\infty\,,$$

and give an example of f for which the inequality turns into equality.

1/I/7C Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed random variables from the $N\left(\mu, \sigma^2\right)$ distribution where μ and σ^2 are unknown. Use the generalized likelihood-ratio test to derive the form of a test of the hypothesis $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$.

Explain carefully how the test should be implemented.

1/II/18C Statistics

Let $X_1, \ldots, X_n$ be independent, identically distributed random variables with

$$\mathbb{P}\left(X_i = 1\right) = \theta = 1 - \mathbb{P}\left(X_i = 0\right) ,$$

where θ is an unknown parameter, $0 < \theta < 1$, and $n \geqslant 2$. It is desired to estimate the quantity $\phi = \theta(1-\theta) = n\mathbb{V}\mathrm{ar}\left(\left(X_1 + \cdots + X_n\right)/n\right)$.

(i) Find the maximum-likelihood estimate, $\hat{\phi}$, of ϕ.

(ii) Show that $\hat{\phi}_1 = X_1\left(1 - X_2\right)$ is an unbiased estimate of ϕ and hence, or otherwise, obtain an unbiased estimate of ϕ which has smaller variance than $\hat{\phi}_1$ and which is a function of $\hat{\phi}$.

(iii) Now suppose that a Bayesian approach is adopted and that the prior distribution for θ, $\pi(\theta)$, is taken to be the uniform distribution on $(0,1)$. Compute the Bayes point estimate of ϕ when the loss function is $L(\phi, a) = (\phi - a)^2$.

[*You may use that fact that when r, s are non-negative integers,*

$$\int_0^1 x^r(1-x)^s dx = r!s!/(r+s+1)! \quad]$$

2/II/19C Statistics

State and prove the Neyman–Pearson lemma.

Suppose that X is a random variable drawn from the probability density function

$$f(x \mid \theta) = \tfrac{1}{2}|\,x\,|^{\theta-1}e^{-|x|}/\Gamma(\theta), \quad -\infty < x < \infty\,,$$

where $\Gamma(\theta) = \int\limits_0^\infty y^{\theta-1}e^{-y}dy$ and $\theta \geqslant 1$ is unknown. Find the most powerful test of size α, $0 < \alpha < 1$, of the hypothesis $H_0 : \theta = 1$ against the alternative $H_1 : \theta = 2$. Express the power of the test as a function of α.

Is your test uniformly most powerful for testing $H_0 : \theta = 1$ against $H_1 : \theta > 1$? Explain your answer carefully.

3/I/8C Statistics

Light bulbs are sold in packets of 3 but some of the bulbs are defective. A sample of 256 packets yields the following figures for the number of defectives in a packet:

No. of defectives	0	1	2	3
No. of packets	116	94	40	6

Test the hypothesis that each bulb has a constant (but unknown) probability θ of being defective independently of all other bulbs.

[*Hint: You may wish to use some of the following percentage points:*

Distribution	χ_1^2	χ_2^2	χ_3^2	χ_4^2	t_1	t_2	t_3	t_4
90% *percentile*	2·71	4·61	6·25	7·78	3·08	1·89	1·64	1·53
95% *percentile*	3·84	5·99	7·81	9·49	6·31	2·92	2·35	2·13

]

4/II/19C Statistics

Consider the linear regression model

$$Y_i = \alpha + \beta x_i + \epsilon_i, \qquad 1 \leqslant i \leqslant n\,,$$

where $\epsilon_1, \ldots, \epsilon_n$ are independent, identically distributed $N(0, \sigma^2)$, $x_1, \ldots, x_n$ are known real numbers with $\sum_{i=1}^n x_i = 0$ and α, β and σ^2 are unknown.

(i) Find the least-squares estimates $\widehat{\alpha}$ and $\widehat{\beta}$ of α and β, respectively, and explain why in this case they are the same as the maximum-likelihood estimates.

(ii) Determine the maximum-likelihood estimate $\widehat{\sigma}^2$ of σ^2 and find a multiple of it which is an unbiased estimate of σ^2.

(iii) Determine the joint distribution of $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\sigma}^2$.

(iv) Explain carefully how you would test the hypothesis $H_0 : \alpha = \alpha_0$ against the alternative $H_1 : \alpha \neq \alpha_0$.

1/I/8C **Optimization**

State and prove the max-flow min-cut theorem for network flows.

2/I/9C **Optimization**

Consider the game with payoff matrix

$$\begin{pmatrix} 2 & 5 & 4 \\ 3 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix},$$

where the (i, j) entry is the payoff to the row player if the row player chooses row i and the column player chooses column j.

Find the value of the game and the optimal strategies for each player.

3/II/20C **Optimization**

State and prove the Lagrangian sufficiency theorem.

Solve the problem

$$\begin{array}{ll} \text{maximize} & x_1 + 3\ln(1 + x_2) \\ \text{subject to} & 2x_1 + 3x_2 \leqslant c_1, \\ & \ln(1 + x_1) \geqslant c_2, \quad x_1 \geqslant 0,\ x_2 \geqslant 0, \end{array}$$

where c_1 and c_2 are non-negative constants satisfying $c_1 + 2 \geqslant 2e^{c_2}$.

4/II/20C **Optimization**

Consider the linear programming problem

$$\begin{array}{ll} \text{minimize} & 2\,x_1 - 3\,x_2 - 2\,x_3 \\ \text{subject to} & -\,2\,x_1 + 2\,x_2 + 4\,x_3 \leqslant 5 \\ & 4\,x_1 + 2\,x_2 - 5\,x_3 \leqslant 8 \\ & 5\,x_1 - 4\,x_2 + \tfrac{1}{2}\,x_3 \leqslant 5\,, \qquad x_i \geqslant 0\,, \quad i = 1, 2, 3\,. \end{array}$$

(i) After adding slack variables z_1, z_2 and z_3 and performing one iteration of the simplex algorithm, the following tableau is obtained.

	x_1	x_2	x_3	z_1	z_2	z_3	
x_2	-1	1	2	$1/2$	0	0	$5/2$
z_2	6	0	-9	-1	1	0	3
z_3	1	0	$17/2$	2	0	1	15
Payoff	-1	0	4	$3/2$	0	0	$15/2$

Complete the solution of the problem.

(ii) Now suppose that the problem is amended so that the objective function becomes

$$2\,x_1 - 3\,x_2 - 5\,x_3\,.$$

Find the solution of this new problem.

(iii) Formulate the dual of the problem in (ii) and identify the optimal solution to the dual.

1/II/19C **Markov Chains**

Consider a Markov chain $(X_n)_{n\geqslant 0}$ on states $\{0,1,\ldots,r\}$ with transition matrix (P_{ij}), where $P_{0,0}=1=P_{r,r}$, so that 0 and r are absorbing states. Let

$$A=(X_n=0,\ \text{for some } n\geqslant 0)\ ,$$

be the event that the chain is absorbed in 0. Assume that $h_i=\mathbb{P}\left(A \mid X_0=i\right)>0$ for $1\leqslant i<r$.

Show carefully that, conditional on the event A, $(X_n)_{n\geqslant 0}$ is a Markov chain and determine its transition matrix.

Now consider the case where $P_{i,i+1}=\frac{1}{2}=P_{i,i-1}$, for $1\leqslant i<r$. Suppose that $X_0=i$, $1\leqslant i<r$, and that the event A occurs; calculate the expected number of transitions until the chain is first in the state 0.

2/II/20C **Markov Chains**

Consider a Markov chain with state space $S=\{0,1,2,\ldots\}$ and transition matrix given by

$$P_{i,j}=\begin{cases} qp^{j-i+1} & \text{for } i\geqslant 1 \text{ and } j\geqslant i-1,\\ qp^{j} & \text{for } i=0 \text{ and } j\geqslant 0,\end{cases}$$

and $P_{i,j}=0$ otherwise, where $0<p=1-q<1$.

For each value of p, $0<p<1$, determine whether the chain is transient, null recurrent or positive recurrent, and in the last case find the invariant distribution.

3/I/9C **Markov Chains**

Consider a Markov chain $(X_n)_{n\geqslant 0}$ with state space $S=\{0,1\}$ and transition matrix

$$P=\begin{pmatrix}\alpha & 1-\alpha\\ 1-\beta & \beta\end{pmatrix},$$

where $0<\alpha<1$ and $0<\beta<1$.

Calculate $\mathbb{P}\left(X_n=0 \mid X_0=0\right)$ for each $n\geqslant 0$.

4/I/9C **Markov Chains**

For a Markov chain with state space S, define what is meant by the following:

(i) states $i,j\in S$ *communicate*;

(ii) state $i\in S$ is *recurrent*.

Prove that communication is an equivalence relation on S and that if two states i,j communicate and i is recurrent then j is recurrent.