UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 9th June, 2006 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Linear Algebra

Suppose V is a vector space over a field k. A finite set of vectors is said to be a *basis* for V if it is both linearly independent and spanning. Prove that any two finite bases for V have the same number of elements.

2E Groups, Rings and Modules

How many elements does the ring $\mathbb{Z}[X]/(3, X^2 + X + 1)$ have?

Is this ring an integral domain?

Briefly justify your answers.

3F Analysis II

Let V be the vector space of all sequences $(x_1, x_2, \ldots)$ of real numbers such that x_i converges to zero. Show that the function

$$|(x_1, x_2, \ldots)| = \max_{i \geqslant 1} |x_i|$$

defines a norm on V.

Is the sequence

$$(1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots), \ldots$$

convergent in V? Justify your answer.

4H Complex Analysis

State the principle of isolated zeros for an analytic function on a domain in $\mathbf{C}$.

Suppose f is an analytic function on $\mathbf{C} \setminus \{0\}$, which is real-valued at the points $1/n$, for $n = 1, 2, \ldots$, and does not have an essential singularity at the origin. Prove that $f(z) = \overline{f(\bar{z})}$ for all $z \in \mathbf{C} \setminus \{0\}$.

5G Methods

A finite-valued function $f(r,\theta,\phi)$, where r,θ,ϕ are spherical polar coordinates, satisfies Laplace's equation in the regions $r<1$ and $r>1$, and $f\to 0$ as $r\to\infty$. At $r=1$, f is continuous and its derivative with respect to r is discontinuous by $A\sin^2\theta$, where A is a constant. Write down the general axisymmetric solution for f in the two regions and use the boundary conditions to find f.

$$\left[\textit{Hint}:\quad P_2(\cos\theta)=\frac{1}{2}\left(3\cos^2\theta-1\right)\,.\right]$$

6B Quantum Mechanics

(a) Define the probability density $\rho(\mathbf{x},t)$ and the probability current $\mathbf{J}(\mathbf{x},t)$ for a quantum mechanical wave function $\psi(\mathbf{x},t)$, where the three dimensional vector $\mathbf{x}$ defines spatial coordinates.

Given that the potential $V(\mathbf{x})$ is real, show that

$$\boldsymbol{\nabla}\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0\,.$$

(b) Write down the standard integral expressions for the expectation value $\langle A\rangle_\psi$ and the uncertainty $\Delta_\psi A$ of a quantum mechanical observable A in a state with wavefunction $\psi(\mathbf{x})$. Give an expression for $\Delta_\psi A$ in terms of $\langle A^2\rangle_\psi$ and $\langle A\rangle_\psi$, and justify your answer.

7G Electromagnetism

Starting from Maxwell's equations, deduce Faraday's law of induction

$$\frac{d\Phi}{dt}=-\varepsilon,$$

for a moving circuit C, where Φ is the flux of $\mathbf{B}$ through the circuit and where the EMF ε is defined to be

$$\varepsilon=\oint_C(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{r}$$

with $\mathbf{v}(\mathbf{r})$ denoting the velocity of a point $\mathbf{r}$ of C.

[*Hint: consider the closed surface consisting of the surface $S(t)$ bounded by $C(t)$, the surface $S(t+\delta t)$ bounded by $C(t+\delta t)$ and the surface S' stretching from $C(t)$ to $C(t+\delta t)$. Show that the flux of* $\mathbf{B}$ *through S' is* $-\oint_C\mathbf{B}\cdot(\mathbf{v}\times d\mathbf{r})\delta t$.]

8D Numerical Analysis

(a) Given the data

x_i	-1	0	1	3
$f(x_i)$	-7	-3	-3	9

find the interpolating cubic polynomial $p \in \mathcal{P}_3$ in the Newton form, and transform it to the power form.

(b) We add to the data one more value $f(x_i)$ at $x_i = 2$. Find the power form of the interpolating quartic polynomial $q \in \mathcal{P}_4$ to the extended data

x_i	-1	0	1	2	3
$f(x_i)$	-7	-3	-3	-7	9

9C Markov Chains

A game of chance is played as follows. At each turn the player tosses a coin, which lands heads or tails with equal probability 1/2. The outcome determines a score for that turn, which depends also on the cumulative score so far. Write S_n for the cumulative score after n turns. In particular $S_0 = 0$. When S_n is odd, a head scores 1 but a tail scores 0. When S_n is a multiple of 4, a head scores 4 and a tail scores 1. When S_n is even but is not a multiple of 4, a head scores 2 and a tail scores 1. By considering a suitable four-state Markov chain, determine the long run proportion of turns for which S_n is a multiple of 4. State clearly any general theorems to which you appeal.

SECTION II

10E Linear Algebra

Suppose that α is an orthogonal endomorphism of the finite-dimensional real inner product space V. Suppose that V is decomposed as a direct sum of mutually orthogonal α-invariant subspaces. How small can these subspaces be made, and how does α act on them? Justify your answer.

Describe the possible matrices for α with respect to a suitably chosen orthonormal basis of V when $\dim V = 3$.

11E Groups, Rings and Modules

(a) Suppose that R is a commutative ring, M an R-module generated by $m_1, \ldots, m_n$ and $\phi \in End_R(M)$. Show that, if $A = (a_{ij})$ is an $n \times n$ matrix with entries in R that represents ϕ with respect to this generating set, then in the sub-ring $R[\phi]$ of $End_R(M)$ we have $\det(a_{ij} - \phi\delta_{ij}) = 0$.

[*Hint: A is a matrix such that $\phi(m_i) = \sum a_{ij} m_j$ with $a_{ij} \in R$. Consider the matrix $C = (a_{ij} - \phi\delta_{ij})$ with entries in $R[\phi]$ and use the fact that for any $n \times n$ matrix N over any commutative ring, there is a matrix N' such that $N'N = (\det N)1_n$.*]

(b) Suppose that k is a field, V a finite-dimensional k-vector space and that $\phi \in End_k(V)$. Show that if A is the matrix of ϕ with respect to some basis of V then ϕ satisfies the characteristic equation $\det(A - \lambda 1) = 0$ of A.

12H Geometry

Describe the hyperbolic lines in both the disc and upper half-plane models of the hyperbolic plane. Given a hyperbolic line l and a point $P \notin l$, we define

$$d(P,l) := \inf_{Q \in l} \rho(P,Q),$$

where ρ denotes the hyperbolic distance. Show that $d(P,l) = \rho(P,Q')$, where Q' is the unique point of l for which the hyperbolic line segment PQ' is perpendicular to l.

Suppose now that L_1 is the positive imaginary axis in the upper half-plane model of the hyperbolic plane, and L_2 is the semicircle with centre $a > 0$ on the real line, and radius r, where $0 < r < a$. For any $P \in L_2$, show that the hyperbolic line through P which is perpendicular to L_1 is a semicircle centred on the origin of radius $\leqslant a + r$, and prove that

$$d(P, L_1) \geqslant \frac{a-r}{a+r}.$$

For arbitrary hyperbolic lines L_1, L_2 in the hyperbolic plane, we define

$$d(L_1, L_2) := \inf_{P \in L_1, Q \in L_2} \rho(P,Q).$$

If L_1 and L_2 are *ultraparallel* (i.e. hyperbolic lines which do not meet, either inside the hyperbolic plane or at its boundary), prove that $d(L_1, L_2)$ is strictly positive.

[*The equivalence of the disc and upper half-plane models of the hyperbolic plane, and standard facts about the metric and isometries of these models, may be quoted without proof.*]

13F Analysis II

State precisely the contraction mapping theorem.

An ancient way to approximate the square root of a positive number a is to start with a guess $x > 0$ and then hope that the average of x and a/x gives a better guess. We can then repeat the procedure using the new guess. Justify this procedure as follows. First, show that all the guesses after the first one are greater than or equal to $\sqrt{a}$. Then apply the properties of contraction mappings to the interval $[\sqrt{a}, \infty)$ to show that the procedure always converges to $\sqrt{a}$.

Once the above procedure is close enough to $\sqrt{a}$, estimate how many more steps of the procedure are needed to get one more decimal digit of accuracy in computing $\sqrt{a}$.

14F Metric and Topological Spaces

(a) Show that every compact subset of a Hausdorff topological space is closed.

(b) Let X be a compact metric space. For F a closed subset of X and p any point of X, show that there is a point q in F with

$$d(p,q) = \inf_{q' \in F} d(p,q').$$

Suppose that for every x and y in X there is a point m in X with $d(x,m) = (1/2)d(x,y)$ and $d(y,m) = (1/2)d(x,y)$. Show that X is connected.

15D Complex Methods

Denote by $f * g$ the convolution of two functions, and by $\widehat{f}$ the Fourier transform, i.e.,

$$[f * g](x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt, \qquad \widehat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,dx\,.$$

(a) Show that, for suitable functions f and g, the Fourier transform $\widehat{F}$ of the convolution $F = f * g$ is given by $\widehat{F} = \widehat{f} \cdot \widehat{g}$.

(b) Let

$$f_1(x) = \begin{cases} 1 & |x| \leqslant 1/2\,, \\ 0 & \text{otherwise.} \end{cases}$$

and let $f_2 = f_1 * f_1$ be the convolution of f_1 with itself. Find the Fourier transforms of f_1 and f_2, and, by applying Parseval's theorem, determine the value of the integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin y}{y}\right)^4 dy\,.$$

16B Methods

The integral

$$I = \int_a^b F(y(x), y'(x))dx\,,$$

where F is some functional, is defined for the class of functions $y(x)$ for which $y(a) = y_0$, with the value $y(b)$ at the upper endpoint unconstrained. Suppose that $y(x)$ extremises the integral among the functions in this class. By considering perturbed paths of the form $y(x) + \epsilon\eta(x)$, with $\epsilon \ll 1$, show that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

and that

$$\left.\frac{\partial F}{\partial y'}\right|_{x=b} = 0\,.$$

Show further that

$$F - y'\frac{\partial F}{\partial y'} = k$$

for some constant k.

A bead slides along a frictionless wire under gravity. The wire lies in a vertical plane with coordinates (x, y) and connects the point A with coordinates $(0, 0)$ to the point B with coordinates $(x_0, y(x_0))$, where x_0 is given and $y(x_0)$ can take any value less than zero. The bead is released from rest at A and slides to B in a time T. For a prescribed x_0 find both the shape of the wire, and the value of $y(x_0)$, for which T is as small as possible.

17B Special Relativity

A javelin of length 4 metres is thrown at a speed of $\frac{12}{13}c$ horizontally and lengthwise through a barn of length 3 metres, which is open at both ends. (Here c denotes the speed of light.)

(a) What is the length of the javelin in the rest frame of the barn?

(b) What is the length of the barn in the rest frame of the javelin?

(c) Define the rest frame coordinates of the barn and of the javelin such that the point where the trailing end of the javelin enters the barn is the origin in both frames. Draw a space-time diagram in the rest frame coordinates (ct, x) of the barn, showing the world lines of both ends of the javelin and of the front and back of the barn. Draw a second space-time diagram in the rest frame coordinates (ct', x') of the javelin, again showing the world lines of both ends of the javelin and of the front and back of the barn.

(d) Clearly mark the space-time events corresponding to (A) the trailing end of the javelin entering the barn, and (B) the leading end of the javelin exiting the back of the barn. Give the corresponding (ct, x) and (ct', x') coordinates for (B).

Are the events (A) and (B) space-like, null, or time-like separated?

As the javelin is longer than the barn in one frame and shorter than the barn in another, it might be argued that the javelin is contained entirely within the barn for a period according to an observer in one frame, but not according to an observer in another. Explain how this apparent inconsistency is resolved.

18A Fluid Dynamics

A rectangular tank has a horizontal base and vertical sides. Viewed from above, the cross-section of the tank is a square of side a. At rest, the depth of water in the tank is h. Suppose that the free-surface is disturbed in such a way that the flow in the water is irrotational. Take the pressure at the free surface as atmospheric. Starting from the appropriate non-linear expressions, obtain free-surface boundary conditions for the velocity potential appropriate for small-amplitude disturbances of the surface.

Show that the governing equations and boundary conditions admit small-amplitude normal mode solutions for which the free-surface elevation above its equilibrium level is everywhere proportional to $e^{i\omega t}$, and find the frequencies, ω, of such modes.

19C Statistics

Two series of experiments are performed, the first resulting in observations $X_1, \ldots, X_m$, the second resulting in observations $Y_1, \ldots, Y_n$. We assume that all observations are independent and normally distributed, with unknown means μ_X in the first series and μ_Y in the second series. We assume further that the variances of the observations are unknown but are all equal.

Write down the distributions of the sample mean $\bar{X} = m^{-1} \sum_{i=1}^{m} X_i$ and sum of squares $S_{XX} = \sum_{i=1}^{m} (X_i - \bar{X})^2$.

Hence obtain a statistic $T(X, Y)$ to test the hypothesis $H_0 : \mu_X = \mu_Y$ against $H_1 : \mu_X > \mu_Y$ and derive its distribution under H_0. Explain how you would carry out a test of size $\alpha = 1/100$.

20C Optimization

Use a suitable version of the simplex algorithm to solve the following linear programming problem:

$$
\begin{array}{lrcrcrcl}
\text{maximize} & 50x_1 & - & 30x_2 & + & x_3 & & \\
\text{subject to} & x_1 & + & x_2 & + & x_3 & \leq & 30 \\
 & 2x_1 & - & x_2 & & & \leq & 35 \\
 & x_1 & + & 2x_2 & - & x_3 & \geq & 40 \\
\text{and} & & & x_1, x_2, x_3 & & & \geq & 0.
\end{array}
$$

END OF PAPER