UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part IB

Tuesday 7 June 2005 9 to 12

## PAPER 1

### Before you begin read these instructions carefully.

*Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most* ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

***Complete answers are preferred to fragments.***

*Write on* ***one side*** *of the paper only and begin each answer on a separate sheet.*

*Write legibly; otherwise, you place yourself at a grave disadvantage.*

***At the end of the examination:***

*Tie up your answers in separate bundles labelled* ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

*Attach a completed gold cover sheet to each bundle; write the examiner letter in the box marked 'Examiner Letter' on the cover sheet.*

*You must also complete a green master cover sheet listing all the questions you have attempted.*

***Every cover sheet must bear your examination number and desk number.***

***STATIONERY REQUIRMENTS***
*Gold cover sheet*
*Green master cover sheet*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

## SECTION I

### 1C Linear Algebra

Let $V$ be an $n$-dimensional vector space over $\mathbf{R}$, and let $\beta : V \to V$ be a linear map. Define the minimal polynomial of $\beta$. Prove that $\beta$ is invertible if and only if the constant term of the minimal polynomial of $\beta$ is non-zero.

### 2A Geometry

Let $\sigma : \mathbf{R}^2 \to \mathbf{R}^3$ be the map defined by

$$\sigma(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u),$$

where $0 < b < a$. Describe briefly the image $T = \sigma(\mathbf{R}^2) \subset \mathbf{R}^3$. Let $V$ denote the open subset of $\mathbf{R}^2$ given by $0 < u < 2\pi$, $0 < v < 2\pi$; prove that the restriction $\sigma|_V$ defines a smooth parametrization of a certain open subset (which you should specify) of $T$. Hence, or otherwise, prove that $T$ is a smooth embedded surface in $\mathbf{R}^3$.

[*You may assume that the image under $\sigma$ of any open set $B \subset \mathbf{R}^2$ is open in $T$.*]

### 3F Complex Analysis or Complex Methods

State the Cauchy integral formula.

Using the Cauchy integral formula, evaluate

$$\int_{|z|=2} \frac{z^3}{z^2+1} dz.$$

### 4G Special Relativity

The four-velocity $U_\mu$ of a particle of rest mass $m$ is defined by $U_\mu = dx_\mu/d\tau$, where $\tau$ is the proper time (the time as measured in the particle's rest frame). Derive the expression for each of the four components of $U_\mu$ in terms of the components of the three-velocity and the speed of light, $c$.

Show that $U \cdot U = c^2$ for an appropriately defined scalar product.

The four-momentum, $p_\mu = m\,U_\mu$, of a particle of rest mass $m$ defines a relativistic generalisation of energy and momentum. Show that the standard non-relativistic expressions for the momentum and kinetic energy of a particle with mass $m$ travelling with velocity $v$ are obtained in the limit $v/c \ll 1$. Show also how the concept of a rest energy equal to $mc^2$ emerges.

**5E Fluid Dynamics**

Explain how a streamfunction $\psi$ can be used to represent in Cartesian Coordinates an incompressible flow in two dimensions. Show that the streamlines are given by $\psi = \text{const}$.

Consider the two-dimensional incompressible flow

$$\mathbf{u}(x, y, t) = (x + \sin t, -y).$$

(a) Find the streamfunction, and hence the streamlines at $t = \dfrac{\pi}{2}$.

(b) Find the path of a fluid particle released at $t = 0$ from $(x_0, 1)$. For what value of $x_0$ does the particle *not* tend to infinity?

**6F Numerical Analysis**

Determine the Cholesky factorization (without pivoting) of the matrix

$$A = \begin{bmatrix} 2 & -4 & 2 \\ -4 & 10+\lambda & 2+3\lambda \\ 2 & 2+3\lambda & 23+9\lambda \end{bmatrix}$$

where $\lambda$ is a real parameter. Hence, find the range of values of $\lambda$ for which the matrix $A$ is positive definite.

**7D Statistics**

The fast-food chain McGonagles have three sizes of their takeaway haggis, Large, Jumbo and Soopersize. A survey of 300 randomly selected customers at one branch choose 92 Large, 89 Jumbo and 119 Soopersize haggises.

Is there sufficient evidence to reject the hypothesis that all three sizes are equally popular? Explain your answer carefully.

| *Distribution* | $t_1$ | $t_2$ | $t_3$ | $\chi_1^2$ | $\chi_2^2$ | $\chi_3^2$ | $F_{1,2}$ | $F_{2,3}$ |
|---|---|---|---|---|---|---|---|---|
| *95% percentile* | 6·31 | 2·92 | 2·35 | 3·84 | 5·99 | 7·82 | 18·51 | 9·55 |

**8D Optimization**

Consider the problem:

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m, \\ & \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n, \\ & x_{ij} \geqslant 0, \quad \text{for all } i, j, \end{aligned}$$

where $a_i \geqslant 0$, $b_j \geqslant 0$ satisfy $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.

Formulate the dual of this problem and state necessary and sufficient conditions for optimality.

## SECTION II

### 9C Linear Algebra

Let $V$ be a finite dimensional vector space over $\mathbf{R}$, and $V^*$ be the dual space of $V$. If $W$ is a subspace of $V$, we define the subspace $\alpha(W)$ of $V^*$ by

$$\alpha(W) = \{f \in V^* : f(w) = 0 \text{ for all } w \text{ in } W\}\,.$$

Prove that $\dim(\alpha(W)) = \dim(V) - \dim(W)$. Deduce that, if $A = (a_{ij})$ is any real $m \times n$-matrix of rank $r$, the equations

$$\sum_{j=1}^{n} a_{ij}\, x_j = 0 \quad (i = 1, \ldots, m)$$

have $n - r$ linearly independent solutions in $\mathbf{R}^n$.

### 10C Groups, Rings and Modules

Let $G$ be a group, and $H$ a subgroup of finite index. By considering an appropriate action of $G$ on the set of left cosets of $H$, prove that $H$ always contains a normal subgroup $K$ of $G$ such that the index of $K$ in $G$ is finite and divides $n!$, where $n$ is the index of $H$ in $G$.

Now assume that $G$ is a finite group of order $pq$, where $p$ and $q$ are prime numbers with $p < q$. Prove that the subgroup of $G$ generated by any element of order $q$ is necessarily normal.

### 11B Analysis II

Let $(f_n)_{n\geqslant 1}$ be a sequence of continuous real-valued functions defined on a set $E \subset \mathbf{R}$. Suppose that the functions $f_n$ converge uniformly to a function $f$. Prove that $f$ is continuous on $E$.

Show that the series $\sum_{n=1}^{\infty} 1/n^{1+x}$ defines a continuous function on the half-open interval $(0, 1]$.

[*Hint: You may assume the convergence of standard series.*]

**12A Metric and Topological Spaces**

Suppose that $(X, d_X)$ and $(Y, d_Y)$ are metric spaces. Show that the definition

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

defines a metric on the product $X \times Y$, under which the projection map $\pi : X \times Y \to Y$ is continuous.

If $(X, d_X)$ is compact, show that every sequence in $X$ has a subsequence converging to a point of $X$. Deduce that the projection map $\pi$ then has the property that, for any closed subset $F \subset X \times Y$, the image $\pi(F)$ is closed in $Y$. Give an example to show that this fails if $(X, d_X)$ is not assumed compact.

**13F Complex Analysis or Complex Methods**

Determine a conformal mapping from $\Omega_0 = \mathbf{C} \setminus [-1, 1]$ to the complex unit disc $\Omega_1 = \{z \in \mathbf{C} : |z| < 1\}$.

[*Hint: A standard method is first to map* $\Omega_0$ *to* $\mathbf{C} \setminus (-\infty, 0]$, *then to the complex right half-plane* $\{z \in \mathbf{C} : \operatorname{Re} z > 0\}$ *and, finally, to* $\Omega_1$.]

**14E Methods**

Find the Fourier Series of the function

$$f(\theta) = \begin{cases} 1 & 0 \le \theta < \pi, \\ -1 & \pi \le \theta < 2\pi. \end{cases}$$

Find the solution $\phi(r, \theta)$ of the Poisson equation in two dimensions inside the unit disk $r \le 1$

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = f(\theta),$$

subject to the boundary condition $\phi(1, \theta) = 0$.

[*Hint: The general solution of* $r^2 R'' + rR' - n^2 R = r^2$ *is* $R = ar^n + br^{-n} - r^2/(n^2 - 4)$. ]

From the solution, show that

$$\int_{r \le 1} f \phi \, dA = -\frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2 (n+2)^2}.$$

**15G Quantum Mechanics**

The wave function of a particle of mass $m$ that moves in a one-dimensional potential well satisfies the Schrödinger equation with a potential that is zero in the region $-a \leq x \leq a$ and infinite elsewhere,

$$V(x) = 0 \quad \text{for} \quad |x| \leq a\,, \qquad V(x) = \infty \quad \text{for} \quad |x| > a\,.$$

Determine the complete set of normalised energy eigenfunctions for the particle and show that the energy eigenvalues are

$$E = \frac{\hbar^2 \pi^2 n^2}{8ma^2}\,,$$

where $n$ is a positive integer.

At time $t = 0$ the wave function is

$$\psi(x) = \frac{1}{\sqrt{5a}} \cos\left(\frac{\pi x}{2a}\right) + \frac{2}{\sqrt{5a}} \sin\left(\frac{\pi x}{a}\right)\,,$$

in the region $-a \leq x \leq a$, and zero otherwise. Determine the possible results for a measurement of the energy of the system and the relative probabilities of obtaining these energies.

In an experiment the system is measured to be in its lowest possible energy eigenstate. The width of the well is then doubled while the wave function is unaltered. Calculate the probability that a later measurement will find the particle to be in the lowest energy state of the new potential well.

**16H Electromagnetism**

For a static charge density $\rho(\mathbf{x})$ show that the energy may be expressed as

$$E = \frac{1}{2} \int \rho \phi \, \mathrm{d}^3 x = \frac{\epsilon_0}{2} \int \mathbf{E}^2 \, \mathrm{d}^3 x\,,$$

where $\phi(\mathbf{x})$ is the electrostatic potential and $\mathbf{E}(\mathbf{x})$ is the electric field.

Determine the scalar potential and electric field for a sphere of radius $R$ with a constant charge density $\rho$. Also determine the total electrostatic energy.

In a nucleus with $Z$ protons the volume is proportional to $Z$. Show that we may expect the electric contribution to energy to be proportional to $Z^{\frac{5}{3}}$.

**17E Fluid Dynamics**

State Bernoulli's expression for the pressure in an unsteady potential flow with conservative force $-\nabla\chi$.

A spherical bubble in an incompressible liquid of density $\rho$ has radius $R(t)$. If the pressure far from the bubble is $p_\infty$ and inside the bubble is $p_b$, show that

$$p_b - p_\infty = \rho\left(\frac{3}{2}\dot{R}^2 + R\ddot{R}\right).$$

Calculate the kinetic energy $K(t)$ in the flow outside the bubble, and hence show that

$$\dot{K} = (p_b - p_\infty)\dot{V},$$

where $V(t)$ is the volume of the bubble.

If $p_b(t) = p_\infty V_0/V$, show that

$$K = K_0 + p_\infty\left(V_0 \ln\frac{V}{V_0} - V + V_0\right),$$

where $K = K_0$ when $V = V_0$.

**18D Statistics**

In the context of hypothesis testing define the following terms: (i) simple hypothesis; (ii) critical region; (iii) size; (iv) power; and (v) type II error probability.

State, without proof, the Neyman–Pearson lemma.

Let $X$ be a single observation from a probability density function $f$. It is desired to test the hypothesis

$$H_0 : f = f_0 \qquad \text{against} \qquad H_1 : f = f_1,$$

with $f_0(x) = \frac{1}{2}|x|\, e^{-x^2/2}$ and $f_1(x) = \Phi'(x)$, $-\infty < x < \infty$, where $\Phi(x)$ is the distribution function of the standard normal, $N(0,1)$.

Determine the best test of size $\alpha$, where $0 < \alpha < 1$, and express its power in terms of $\Phi$ and $\alpha$.

Find the size of the test that minimizes the sum of the error probabilities. Explain your reasoning carefully.

**19D Markov Chains**

Every night Lancelot and Guinevere sit down with four guests for a meal at a circular dining table. The six diners are equally spaced around the table and just before each meal two individuals are chosen at random and they exchange places from the previous night while the other four diners stay in the same places they occupied at the last meal; the choices on successive nights are made independently. On the first night Lancelot and Guinevere are seated next to each other.

Find the probability that they are seated diametrically opposite each other on the $(n+1)$th night at the round table, $n \geqslant 1$.

## END OF PAPER