UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday 2 June 2004 1.30 to 4.30

PAPER 2

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. You should attempt at most **four** *questions from Section I and at most* **six** *questions from Section II.*

Additional credit will be given to substantially complete answers.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled **A, B,** . . . , **H** *according to the code letter affixed to each question, including in the same bundle questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your candidate number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

For each n let A_n be the $n \times n$ matrix defined by

$$(A_n)_{ij} = \begin{cases} i & i \leqslant j\,, \\ j & i > j\,. \end{cases}$$

What is $\det A_n$? Justify your answer.

[*It may be helpful to look at the cases* $n = 1, 2, 3$ *before tackling the general case.*]

2F Groups, Rings and Modules

Prove that the alternating group A_5 is simple.

3G Analysis II

Consider a sequence of continuous functions $F_n : [-1, 1] \to \mathbb{R}$. Suppose that the functions F_n converge uniformly to some continuous function F. Show that the integrals $\int_{-1}^{1} F_n(x)dx$ converge to $\int_{-1}^{1} F(x)dx$.

Give an example to show that, even if the functions $F_n(x)$ and $F(x)$ are differentiable, the derivatives $F_n'(0)$ need not converge to $F'(0)$.

4E Further Analysis

Let τ be the topology on $\mathbb{N}$ consisting of the empty set and all sets $X \subset \mathbb{N}$ such that $\mathbb{N} \setminus X$ is finite. Let σ be the usual topology on $\mathbb{R}$, and let ρ be the topology on $\mathbb{R}$ consisting of the empty set and all sets of the form (x, ∞) for some real x.

(i) Prove that all continuous functions $f : (\mathbb{N}, \tau) \to (\mathbb{R}, \sigma)$ are constant.

(ii) Give an example with proof of a non-constant function $f : (\mathbb{N}, \tau) \to (\mathbb{R}, \rho)$ that is continuous.

5A Complex Methods

Let the functions f and g be analytic in an open, nonempty domain Ω and assume that $g \neq 0$ there. Prove that if $|f(z)| \equiv |g(z)|$ in Ω then there exists $\alpha \in \mathbb{R}$ such that $f(z) \equiv e^{i\alpha} g(z)$.

6B Methods

Write down the general form of the solution in polar coordinates (r, θ) to Laplace's equation in two dimensions.

Solve Laplace's equation for $\phi(r, \theta)$ in $0 < r < 1$ and in $1 < r < \infty$, subject to the conditions

$$\phi \to 0 \qquad \text{as} \qquad r \to 0 \text{ and } r \to \infty,$$

$$\phi|_{r=1+} = \phi|_{r=1-} \qquad \text{and} \qquad \left.\frac{\partial \phi}{\partial r}\right|_{r=1+} - \left.\frac{\partial \phi}{\partial r}\right|_{r=1-} = \cos 2\theta + \cos 4\theta.$$

7B Electromagnetism

Write down the two Maxwell equations that govern steady magnetic fields. Show that the boundary conditions satisfied by the magnetic field on either side of a sheet carrying a surface current of density $\mathbf{s}$, with normal $\mathbf{n}$ to the sheet, are

$$\mathbf{n} \times \mathbf{B}_+ - \mathbf{n} \times \mathbf{B}_- = \mu_0 \mathbf{s}.$$

Write down the force per unit area on the surface current.

8D Quantum Mechanics

A quantum mechanical system is described by vectors $\psi = \begin{pmatrix} a \\ b \end{pmatrix}$. The energy eigenvectors are

$$\psi_0 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \psi_1 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

with energies E_0, E_1 respectively. The system is in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at time $t = 0$. What is the probability of finding it in the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at a later time t?

9A Numerical Analysis

Determine the coefficients of Gaussian quadrature for the evaluation of the integral

$$\int_0^1 f(x) x \, dx$$

that uses two function evaluations.

10H Statistics

A study of 60 men and 90 women classified each individual according to eye colour to produce the figures below.

	Blue	Brown	Green
Men	20	20	20
Women	20	50	20

Explain how you would analyse these results. You should indicate carefully any underlying assumptions that you are making.

A further study took 150 individuals and classified them both by eye colour and by whether they were left or right handed to produce the following table.

	Blue	Brown	Green
Left Handed	20	20	20
Right Handed	20	50	20

How would your analysis change? You should again set out your underlying assumptions carefully.

[*You may wish to note the following percentiles of the* χ^2 *distribution.*

	χ_1^2	χ_2^2	χ_3^2	χ_4^2	χ_5^2	χ_6^2
95% percentile	3.84	5.99	7.81	9.49	11.07	12.59
99% percentile	6.64	9.21	11.34	13.28	15.09	16.81

]

11H Markov Chains

Let $(X_r)_{r\geqslant 0}$ be an irreducible, positive-recurrent Markov chain on the state space S with transition matrix (P_{ij}) and initial distribution $P(X_0 = i) = \pi_i$, $i \in S$, where (π_i) is the unique invariant distribution. What does it mean to say that the Markov chain is reversible?

Prove that the Markov chain is reversible if and only if $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i, j \in S$.

SECTION II

12E Linear Algebra

Let Q be a quadratic form on a real vector space V of dimension n. Prove that there is a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ with respect to which Q is given by the formula

$$Q\Big(\sum_{i=1}^{n} x_i \mathbf{e}_i\Big) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 \,.$$

Prove that the numbers p and q are uniquely determined by the form Q. By means of an example, show that the subspaces $\langle \mathbf{e}_1, \ldots, \mathbf{e}_p \rangle$ and $\langle \mathbf{e}_{p+1}, \ldots, \mathbf{e}_{p+q} \rangle$ need not be uniquely determined by Q.

13F Groups, Rings and Modules

Let K be a subgroup of a group G. Prove that K is normal if and only if there is a group H and a homomorphism $\phi : G \to H$ such that

$$K = \{ g \in G \;:\; \phi(g) = 1 \} \,.$$

Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d in $\mathbb{Z}$ and $ad - bc = 1$. Let p be a prime number, and take K to be the subset of G consisting of all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \equiv d \equiv 1 \,(\mathrm{mod}\ p)$ and $c \equiv b \equiv 0 \,(\mathrm{mod}\ p)$. Prove that K is a normal subgroup of G.

14G Analysis II

Let X be a non-empty complete metric space. Give an example to show that the intersection of a descending sequence of non-empty closed subsets of X, $A_1 \supset A_2 \supset \cdots$, can be empty. Show that if we also assume that

$$\lim_{n \to \infty} \mathrm{diam}\ (A_n) = 0$$

then the intersection is not empty. Here the diameter $\mathrm{diam}(A)$ is defined as the supremum of the distances between any two points of a set A.

We say that a subset A of X is *dense* if it has nonempty intersection with every nonempty open subset of X. Let $U_1, U_2, \ldots$ be any sequence of dense open subsets of X. Show that the intersection $\bigcap_{n=1}^{\infty} U_n$ is not empty.

[*Hint: Look for a descending sequence of subsets* $A_1 \supset A_2 \supset \cdots$, *with* $A_i \subset U_i$, *such that the previous part of this problem applies.*]

15E Further Analysis

(i) Let X be the set of all infinite sequences $(\epsilon_1, \epsilon_2, \ldots)$ such that $\epsilon_i \in \{0,1\}$ for all i. Let τ be the collection of all subsets $Y \subset X$ such that, for every $(\epsilon_1, \epsilon_2, \ldots) \in Y$ there exists n such that $(\eta_1, \eta_2, \ldots) \in Y$ whenever $\eta_1 = \epsilon_1, \eta_2 = \epsilon_2, \ldots, \eta_n = \epsilon_n$. Prove that τ is a topology on X.

(ii) Let a distance d be defined on X by

$$d\Big((\epsilon_1, \epsilon_2, \ldots), (\eta_1, \eta_2, \ldots)\Big) = \sum_{n=1}^{\infty} 2^{-n} |\epsilon_n - \eta_n| \,.$$

Prove that d is a metric and that the topology arising from d is the same as τ.

16A Complex Methods

Prove by using the Cauchy theorem that if f is analytic in the open disc $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ then there exists a function g, analytic in Ω, such that $g'(z) = f(z)$, $z \in \Omega$.

17B Methods

Let $I_{ij}(P)$ be the moment-of-inertia tensor of a rigid body relative to the point P. If G is the centre of mass of the body and the vector GP has components X_i, show that

$$I_{ij}(P) = I_{ij}(G) + M\left(X_k X_k \delta_{ij} - X_i X_j\right),$$

where M is the mass of the body.

Consider a cube of uniform density and side $2a$, with centre at the origin. Find the inertia tensor about the centre of mass, and thence about the corner $P = (a, a, a)$.

Find the eigenvectors and eigenvalues of $I_{ij}(P)$.

18B Electromagnetism

The vector potential due to a steady current density $\mathbf{J}$ is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3\mathbf{r}', \qquad (*)$$

where you may assume that $\mathbf{J}$ extends only over a finite region of space. Use $(*)$ to derive the Biot–Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3\mathbf{r}'.$$

A circular loop of wire of radius a carries a current I. Take Cartesian coordinates with the origin at the centre of the loop and the z-axis normal to the loop. Use the Biot–Savart law to show that on the z-axis the magnetic field is in the axial direction and of magnitude

$$B = \frac{\mu_0 I a^2}{2(z^2 + a^2)^{3/2}}.$$

19D Quantum Mechanics

Consider a Hamiltonian of the form

$$H = \frac{1}{2m}\big(p + if(x)\big)\big(p - if(x)\big), \qquad -\infty < x < \infty\,,$$

where $f(x)$ is a real function. Show that this can be written in the form $H = p^2/(2m) + V(x)$, for some real $V(x)$ to be determined. Show that there is a wave function $\psi_0(x)$, satisfying a first-order equation, such that $H\psi_0 = 0$. If f is a polynomial of degree n, show that n must be odd in order for ψ_0 to be normalisable. By considering $\int \mathrm{d}x\, \psi^* H \psi$ show that all energy eigenvalues other than that for ψ_0 must be positive.

For $f(x) = kx$, use these results to find the lowest energy and corresponding wave function for the harmonic oscillator Hamiltonian

$$H_{\text{oscillator}} = \frac{p^2}{2m} + \tfrac{1}{2} m\omega^2 x^2\,.$$

20A Numerical Analysis

Given an $m \times n$ matrix A and $\mathbf{b} \in \mathbb{R}^m$, prove that the vector $\mathbf{x} \in \mathbb{R}^n$ is the solution of the least-squares problem for $A\mathbf{x} \approx \mathbf{b}$ if and only if $A^T(A\mathbf{x} - \mathbf{b}) = \mathbf{0}$. Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & 3 \\ 4 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \end{bmatrix}.$$

Determine the solution of the least-squares problem for $A\mathbf{x} \approx \mathbf{b}$.

21H Statistics

Defining carefully the terminology that you use, state and prove the Neyman–Pearson Lemma.

Let X be a single observation from the distribution with density function

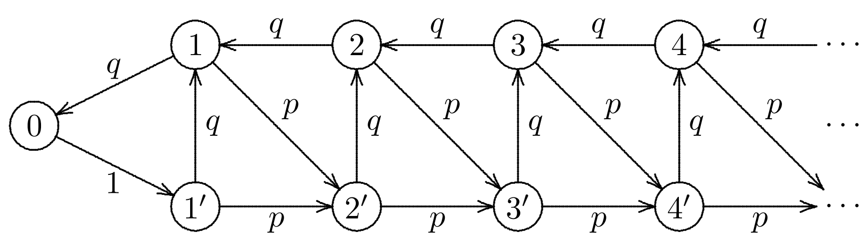
$$f(x \mid \theta) = \tfrac{1}{2} e^{-|x-\theta|}, \quad -\infty < x < \infty,$$

for an unknown real parameter θ. Find the best test of size α, $0 < \alpha < 1$, of the hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, where $\theta_1 > \theta_0$.

When $\alpha = 0.05$, for which values of θ_0 and θ_1 will the power of the best test be at least 0.95?

22H Markov Chains

Consider a Markov chain on the state space $S = \{0, 1, 2, \ldots\} \cup \{1', 2', 3', \ldots\}$ with transition probabilities as illustrated in the diagram below, where $0 < q < 1$ and $p = 1 - q$.



For each value of q, $0 < q < 1$, determine whether the chain is transient, null recurrent or positive recurrent.

When the chain is positive recurrent, calculate the invariant distribution.