UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

List of Courses

1/I/5G **Linear Mathematics**

Define $f : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$f(a,b,c) = (a + 3b - c, 2b + c, -4b - c).$$

Find the characteristic polynomial and the minimal polynomial of f. Is f diagonalisable? Are f and f^2 linearly independent endomorphisms of $\mathbb{C}^3$? Justify your answers.

1/II/14G **Linear Mathematics**

Let α be an endomorphism of a vector space V of finite dimension n .

(a) What is the dimension of the vector space of linear endomorphisms of V? Show that there exists a non-trivial polynomial $p(X)$ such that $p(\alpha) = 0$. Define what is meant by the minimal polynomial m_α of α.

(b) Show that the eigenvalues of α are precisely the roots of the minimal polynomial of α.

(c) Let W be a subspace of V such that $\alpha(W) \subseteq W$ and let β be the restriction of α to W. Show that m_β divides m_α.

(d) Give an example of an endomorphism α and a subspace W as in (c) not equal to V for which $m_\alpha = m_\beta$, and $\deg(m_\alpha) > 1$.

2/I/6G **Linear Mathematics**

Let A be a complex 4×4 matrix such that $A^3 = A^2$. What are the possible minimal polynomials of A? If A is not diagonalisable and $A^2 \neq 0$, list all possible Jordan normal forms of A.

2/II/15G **Linear Mathematics**

(a) A complex $n \times n$ matrix is said to be unipotent if $U - I$ is nilpotent, where I is the identity matrix. Show that U is unipotent if and only if 1 is the only eigenvalue of U.

(b) Let T be an invertible complex matrix. By considering the Jordan normal form of T show that there exists an invertible matrix P such that

$$PTP^{-1} = D_0 + N,$$

where D_0 is an invertible diagonal matrix, N is an upper triangular matrix with zeros in the diagonal and $D_0 N = N D_0$.

(c) Set $D = P^{-1} D_0 P$ and show that $U = D^{-1} T$ is unipotent.

(d) Conclude that any invertible matrix T can be written as $T = DU$ where D is diagonalisable, U is unipotent and $DU = UD$.

3/I/7F Linear Mathematics

Which of the following statements are true, and which false? Give brief justifications for your answers.

(a) If U and W are subspaces of a vector space V, then $U \cap W$ is always a subspace of V.

(b) If U and W are distinct subspaces of a vector space V, then $U \cup W$ is never a subspace of V.

(c) If U, W and X are subspaces of a vector space V, then $U \cap (W + X) = (U \cap W) + (U \cap X)$.

(d) If U is a subspace of a finite-dimensional space V, then there exists a subspace W such that $U \cap W = \{0\}$ and $U + W = V$.

3/II/17F Linear Mathematics

Define the *determinant* of an $n \times n$ matrix A, and prove from your definition that if A' is obtained from A by an elementary row operation (i.e. by adding a scalar multiple of the ith row of A to the jth row, for some $j \neq i$), then $\det A' = \det A$.

Prove also that if X is a $2n \times 2n$ matrix of the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where O denotes the $n \times n$ zero matrix, then $\det X = \det A \det C$. Explain briefly how the $2n \times 2n$ matrix

$$\begin{pmatrix} B & I \\ O & A \end{pmatrix}$$

can be transformed into the matrix

$$\begin{pmatrix} B & I \\ -AB & O \end{pmatrix}$$

by a sequence of elementary row operations. Hence or otherwise prove that $\det AB = \det A \det B$.

4/I/6F Linear Mathematics

Define the *rank* and *nullity* of a linear map between finite-dimensional vector spaces. State the rank–nullity formula.

Let $\alpha\colon U \to V$ and $\beta\colon V \to W$ be linear maps. Prove that

$$\operatorname{rank}(\alpha) + \operatorname{rank}(\beta) - \dim V \leqslant \operatorname{rank}(\beta\alpha) \leqslant \min\{\operatorname{rank}(\alpha), \operatorname{rank}(\beta)\}\ .$$

4/II/15F Linear Mathematics

Define the *dual space* V^* of a finite-dimensional real vector space V, and explain what is meant by the basis of V^* dual to a given basis of V. Explain also what is meant by the statement that the second dual V^{**} is naturally isomorphic to V.

Let V_n denote the space of real polynomials of degree at most n. Show that, for any real number x, the function e_x mapping p to $p(x)$ is an element of V_n^*. Show also that, if $x_1, x_2, \ldots, x_{n+1}$ are distinct real numbers, then $\{e_{x_1}, e_{x_2}, \ldots, e_{x_{n+1}}\}$ is a basis of V_n^*, and find the basis of V_n dual to it.

Deduce that, for any $(n+1)$ distinct points $x_1, \ldots, x_{n+1}$ of the interval $[-1, 1]$, there exist scalars $\lambda_1, \ldots, \lambda_{n+1}$ such that

$$\int_{-1}^{1} p(t)\, dt = \sum_{i=1}^{n+1} \lambda_i p(x_i)$$

for all $p \in V_n$. For $n = 4$ and $(x_1, x_2, x_3, x_4, x_5) = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$, find the corresponding scalars λ_i.

1/I/4E **Geometry**

Show that any finite group of orientation-preserving isometries of the Euclidean plane is cyclic.

Show that any finite group of orientation-preserving isometries of the hyperbolic plane is cyclic.

[*You may assume that given any non-empty finite set E in the hyperbolic plane, or the Euclidean plane, there is a unique smallest closed disc that contains E. You may also use any general fact about the hyperbolic plane without proof providing that it is stated carefully.*]

1/II/13E **Geometry**

Let $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$, and let $\mathbb{H}$ have the hyperbolic metric ρ derived from the line element $|dz|/y$. Let Γ be the group of Möbius maps of the form $z \mapsto (az+b)/(cz+d)$, where a, b, c and d are real and $ad - bc = 1$. Show that every g in Γ is an isometry of the metric space $(\mathbb{H}, \rho)$. For z and w in $\mathbb{H}$, let

$$h(z,w) = \frac{|z-w|^2}{\mathrm{Im}(z)\mathrm{Im}(w)}.$$

Show that for every g in Γ, $h\big(g(z), g(w)\big) = h(z,w)$. By considering $z = iy$, where $y > 1$, and $w = i$, or otherwise, show that for all z and w in $\mathbb{H}$,

$$\cosh \rho(z,w) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\mathrm{Im}(w)}.$$

By considering points i, iy, where $y > 1$ and $s + it$, where $s^2 + t^2 = 1$, or otherwise, derive Pythagoras' Theorem for hyperbolic geometry in the form $\cosh a \cosh b = \cosh c$, where a, b and c are the lengths of sides of a right-angled triangle whose hypotenuse has length c.

3/I/4E **Geometry**

State Euler's formula for a graph $\mathcal{G}$ with F faces, E edges and V vertices on the surface of a sphere.

Suppose that every face in $\mathcal{G}$ has at least three edges, and that at least three edges meet at every vertex of $\mathcal{G}$. Let F_n be the number of faces of $\mathcal{G}$ that have exactly n edges ($n \geqslant 3$), and let V_m be the number of vertices at which exactly m edges meet ($m \geqslant 3$). By expressing $6F - \sum_n nF_n$ in terms of the V_j, or otherwise, show that every convex polyhedron has at least four faces each of which is a triangle, a quadrilateral or a pentagon.

3/II/14E **Geometry**

Show that every isometry of Euclidean space $\mathbb{R}^3$ is a composition of reflections in planes.

What is the smallest integer N such that every isometry f of $\mathbb{R}^3$ with $f(0) = 0$ can be expressed as the composition of at most N reflections? Give an example of an isometry that needs this number of reflections and justify your answer.

Describe (geometrically) all twelve orientation-reversing isometries of a regular tetrahedron.

1/I/1E **Analysis II**

Suppose that for each $n = 1, 2, \ldots$, the function $f_n : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(a) If $f_n \to f$ pointwise on $\mathbb{R}$ is f necessarily continuous on $\mathbb{R}$?

(b) If $f_n \to f$ uniformly on $\mathbb{R}$ is f necessarily continuous on $\mathbb{R}$?

In each case, give a proof or a counter-example (with justification).

1/II/10E **Analysis II**

Suppose that (X, d) is a metric space that has the Bolzano-Weierstrass property (that is, any sequence has a convergent subsequence). Let (Y, d') be any metric space, and suppose that f is a continuous map of X onto Y. Show that (Y, d') also has the Bolzano-Weierstrass property.

Show also that if f is a bijection of X onto Y, then $f^{-1} : Y \to X$ is continuous.

By considering the map $x \mapsto e^{ix}$ defined on the real interval $[-\pi/2, \pi/2]$, or otherwise, show that there exists a continuous choice of $\arg z$ for the complex number z lying in the right half-plane $\{x + iy : x > 0\}$.

2/I/1E **Analysis II**

Define what is meant by (i) a complete metric space, and (ii) a totally bounded metric space.

Give an example of a metric space that is complete but not totally bounded. Give an example of a metric space that is totally bounded but not complete.

Give an example of a continuous function that maps a complete metric space onto a metric space that is not complete. Give an example of a continuous function that maps a totally bounded metric space onto a metric space that is not totally bounded.

[*You need not justify your examples.*]

2/II/10E **Analysis II**

(a) Let f be a map of a complete metric space (X, d) into itself, and suppose that there exists some k in $(0,1)$, and some positive integer N, such that $d\big(f^N(x), f^N(y)\big) \leqslant k\, d(x,y)$ for all distinct x and y in X, where f^m is the mth iterate of f. Show that f has a unique fixed point in X.

(b) Let f be a map of a compact metric space (X, d) into itself such that $d\big(f(x), f(y)\big) < d(x,y)$ for all distinct x and y in X. By considering the function $d\big(f(x), x\big)$, or otherwise, show that f has a unique fixed point in X.

(c) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|f(x) - f(y)| < |x - y|$ for every distinct x and y in $\mathbb{R}^n$. Suppose that for some x, the orbit $O(x) = \{x, f(x), f^2(x), \ldots\}$ is bounded. Show that f maps the closure of $O(x)$ into itself, and deduce that f has a unique fixed point in $\mathbb{R}^n$.

[*The Contraction Mapping Theorem may be used without proof providing that it is correctly stated.*]

3/I/1E **Analysis II**

Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f = (u, v)$, where u and v are defined by $u(0) = v(0) = 0$ and, for $t \neq 0$, $u(t) = t^2 \sin(1/t)$ and $v(t) = t^2 \cos(1/t)$. Show that f is differentiable on $\mathbb{R}$.

Show that for any real non-zero a, $||f'(a) - f'(0)|| > 1$, where we regard $f'(a)$ as the vector $\big(u'(a), v'(a)\big)$ in $\mathbb{R}^2$.

3/II/11E Analysis II

Show that if a, b and c are non-negative numbers, and $a \leqslant b+c$, then

$$\frac{a}{1+a} \leqslant \frac{b}{1+b} + \frac{c}{1+c}.$$

Deduce that if (X,d) is a metric space, then $d(x,y)/[1+d(x,y)]$ is a metric on X.

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and $K_n = \{z \in D : |z| \leqslant (n-1)/n\}$. Let $\mathcal{F}$ be the class of continuous complex-valued functions on D and, for f and g in $\mathcal{F}$, define

$$\sigma(f,g) = \sum_{n=2}^{\infty} \frac{1}{2^n} \frac{||f-g||_n}{1+||f-g||_n},$$

where $||f-g||_n = \sup\{|f(z)-g(z)| : z \in K_n\}$. Show that the series for $\sigma(f,g)$ converges, and that σ is a metric on $\mathcal{F}$.

For $|z|<1$, let $s_k(z) = 1+z+z^2+\cdots+z^k$ and $s(z) = 1+z+z^2+\cdots$. Show that for $n \geqslant 2$, $||s_k - s||_n = n(1-\frac{1}{n})^{k+1}$. By considering the sums for $2 \leqslant n \leqslant N$ and $n > N$ separately, show that for each N,

$$\sigma(s_k, s) \leqslant \sum_{n=2}^{N} ||s_k - s||_n + 2^{-N},$$

and deduce that $\sigma(s_k, s) \to 0$ as $k \to \infty$.

4/I/1E Analysis II

(a) Let (X,d) be a metric space containing the point x_0, and let

$$U = \{x \in X : d(x,x_0) < 1\}, \quad K = \{x \in X : d(x,x_0) \leqslant 1\}.$$

Is U necessarily the largest open subset of K? Is K necessarily the smallest closed set that contains U? Justify your answers.

(b) Let X be a normed space with norm $||\cdot||$, and let

$$U = \{x \in X : ||x|| < 1\}, \quad K = \{x \in X : ||x|| \leqslant 1\}.$$

Is U necessarily the largest open subset of K? Is K necessarily the smallest closed set that contains U? Justify your answers.

4/II/10E **Analysis II**

(a) Let V be a finite-dimensional real vector space, and let $||\cdot||_1$ and $||\cdot||_2$ be two norms on V. Show that a function $f : V \to \mathbb{R}$ is differentiable at a point a in V with respect to $||\cdot||_1$ if and only if it is differentiable at a with respect to $||\cdot||_2$, and that if this is so then the derivative $f'(a)$ of f is independent of the norm used. [*You may assume that all norms on a finite-dimensional vector space are equivalent.*]

(b) Let V_1, V_2 and V_3 be finite-dimensional normed real vector spaces with V_j having norm $||\cdot||_j$, $j = 1, 2, 3$, and let $f : V_1 \times V_2 \to V_3$ be a continuous bilinear mapping. Show that f is differentiable at any point (a, b) in $V_1 \times V_2$, and that $f'(a, b)(h, k) = f(h, b) + f(a, k)$. [*You may assume that* $\left(||u||_1^2 + ||v||_2^2\right)^{1/2}$ *is a norm on* $V_1 \times V_2$, *and that* $\{(x, y) \in V_1 \times V_2 : ||x||_1 = 1, ||y||_2 = 1\}$ *is compact.*]

1/I/7B **Complex Methods**

Using contour integration around a rectangle with vertices

$$-x,\ x,\ x+iy,\ -x+iy,$$

prove that, for all real y,

$$\int_{-\infty}^{+\infty} e^{-(x+iy)^2}\, dx = \int_{-\infty}^{+\infty} e^{-x^2}\, dx\,.$$

Hence derive that the function $f(x) = e^{-x^2/2}$ is an eigenfunction of the Fourier transform

$$\widehat{f}(y) = \int_{-\infty}^{+\infty} f(x) e^{-ixy} dx,$$

i.e. $\widehat{f}$ is a constant multiple of f.

1/II/16B **Complex Methods**

(a) Show that if f is an analytic function at z_0 and $f'(z_0) \neq 0$, then f is conformal at z_0, i.e. it preserves angles between paths passing through z_0.

(b) Let D be the disc given by $|z+i| < \sqrt{2}$, and let H be the half-plane given by $y > 0$, where $z = x+iy$. Construct a map of the domain $D \cap H$ onto H, and hence find a conformal mapping of $D \cap H$ onto the disc $\{z : |z| < 1\}$. [*Hint: You may find it helpful to consider a mapping of the form* $(az+b)/(cz+d)$, *where* $ad - bc \neq 0$.]

2/I/7B **Complex Methods**

Suppose that f is analytic, and that $|f(z)|^2$ is constant in an open disk D. Use the Cauchy–Riemann equations to show that $f(z)$ is constant in D.

2/II/16B **Complex Methods**

A function $f(z)$ has an isolated singularity at a, with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

(a) Define $\operatorname{res}(f, a)$, the residue of f at the point a.

(b) Prove that if a is a pole of order $k+1$, then

$$\operatorname{res}(f,a) = \lim_{z \to a} \frac{h^{(k)}(z)}{k!}, \quad \text{where} \quad h(z) = (z-a)^{k+1} f(z).$$

(c) Using the residue theorem and the formula above show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{k+1}} = \pi \frac{(2k)!}{(k!)^2} 4^{-k}, \quad k \geq 1.$$

4/I/8B **Complex Methods**

Let f be a function such that $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$. Prove that

$$\int_{-\infty}^{+\infty} f(x+k) \overline{f(x+l)}\, dx = 0 \quad \text{for all integers } k \text{ and } l \text{ with } k \neq l,$$

if and only if

$$\int_{-\infty}^{+\infty} |\widehat{f}(t)|^2 e^{-imt} dt = 0 \quad \text{for all integers } m \neq 0,$$

where $\widehat{f}$ is the Fourier transform of f.

4/II/17B **Complex Methods**

(a) Using the inequality $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \frac{\pi}{2}$, show that, if f is continuous for large $|z|$, and if $f(z) \to 0$ as $z \to \infty$, then

$$\lim_{R \to \infty} \int_{\Gamma_R} f(z) e^{i\lambda z} dz = 0 \quad \text{for} \quad \lambda > 0,$$

where $\Gamma_R = Re^{i\theta}$, $0 \leq \theta \leq \pi$.

(b) By integrating an appropriate function $f(z)$ along the contour formed by the semicircles Γ_R and Γ_r in the upper half-plane with the segments of the real axis $[-R, -r]$ and $[r, R]$, show that

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

1/I/2A **Methods**

Find the Fourier sine series for $f(x) = x$, on $0 \leqslant x < L$. To which value does the series converge at $x = \frac{3}{2}L$?

Now consider the corresponding cosine series for $f(x) = x$, on $0 \leqslant x < L$. Sketch the cosine series between $x = -2L$ and $x = 2L$. To which value does the series converge at $x = \frac{3}{2}L$? [*You do not need to determine the cosine series explicitly.*]

1/II/11A **Methods**

The potential $\Phi(r, \vartheta)$, satisfies Laplace's equation everywhere except on a sphere of unit radius and $\Phi \to 0$ as $r \to \infty$. The potential is continuous at $r = 1$, but the derivative of the potential satisfies

$$\lim_{r \to 1^+} \frac{\partial \Phi}{\partial r} - \lim_{r \to 1^-} \frac{\partial \Phi}{\partial r} = V \cos^2 \vartheta,$$

where V is a constant. Use the method of separation of variables to find Φ for both $r > 1$ and $r < 1$.

[*The Laplacian in spherical polar coordinates for axisymmetric systems is*

$$\nabla^2 \equiv \frac{1}{r^2} \left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \vartheta} \left(\frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial}{\partial \vartheta} \right).$$

You may assume that the equation

$$\left((1 - x^2) y' \right)' + \lambda y = 0$$

has polynomial solutions of degree n*, which are regular at* $x = \pm 1$*, if and only if* $\lambda = n(n+1)$*.*]

2/I/2C **Methods**

Write down the transformation law for the components of a second-rank tensor A_{ij} explaining the meaning of the symbols that you use.

A tensor is said to have *cubic symmetry* if its components are unchanged by rotations of $\pi/2$ about each of the three co-ordinate axes. Find the most general second-rank tensor having cubic symmetry.

2/II/11C **Methods**

If $\mathbf{B}$ is a vector, and

$$T_{ij} = \alpha B_i B_j + \beta B_k B_k \delta_{ij} \ ,$$

show for arbitrary scalars α and β that T_{ij} is a symmetric second-rank tensor.

Find the eigenvalues and eigenvectors of T_{ij}.

Suppose now that $\mathbf{B}$ depends upon position $\mathbf{x}$ and that $\nabla \cdot \mathbf{B} = 0$. Find constants α and β such that

$$\frac{\partial}{\partial x_j} T_{ij} = [(\nabla \times \mathbf{B}) \times \mathbf{B}]_i \ .$$

Hence or otherwise show that if $\mathbf{B}$ vanishes everywhere on a surface S that encloses a volume V then

$$\int_V (\nabla \times \mathbf{B}) \times \mathbf{B} \ dV = 0 \ .$$

3/I/2A **Methods**

Write down the wave equation for the displacement $y(x,t)$ of a stretched string with constant mass density and tension. Obtain the general solution in the form

$$y(x,t) = f(x+ct) + g(x-ct),$$

where c is the wave velocity. For a solution in the region $0 \leqslant x < \infty$, with $y(0,t) = 0$ and $y \to 0$ as $x \to \infty$, show that

$$E = \int_0^\infty \left[\frac{1}{2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} c^2 \left(\frac{\partial y}{\partial x} \right)^2 \right] dx,$$

is constant in time. Express E in terms of the general solution in this case.

3/II/12A Methods

Consider the real Sturm-Liouville problem

$$\mathcal{L}y(x) = -(p(x)y')' + q(x)y = \lambda r(x)y,$$

with the boundary conditions $y(a) = y(b) = 0$, where p, q and r are continuous and positive on $[a, b]$. Show that, with suitable choices of inner product and normalisation, the eigenfunctions $y_n(x)$, $n = 1, 2, 3\ldots$, form an orthonormal set.

Hence show that the corresponding Green's function $G(x, \xi)$ satisfying

$$(\mathcal{L} - \mu r(x))G(x, \xi) = \delta(x - \xi),$$

where μ is not an eigenvalue, is

$$G(x, \xi) = \sum_{n=1}^{\infty} \frac{y_n(x)y_n(\xi)}{\lambda_n - \mu},$$

where λ_n is the eigenvalue corresponding to y_n.

Find the Green's function in the case where

$$\mathcal{L}y \equiv y'',$$

with boundary conditions $y(0) = y(\pi) = 0$, and deduce, by suitable choice of μ, that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

4/I/2A Methods

Use the method of Lagrange multipliers to find the largest volume of a rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

4/II/11A Methods

A function $y(x)$ is chosen to make the integral

$$I = \int_a^b f\left(x, y, y', y''\right) dx$$

stationary, subject to given values of $y(a), y'(a), y(b)$ and $y'(b)$. Derive an analogue of the Euler–Lagrange equation for $y(x)$.

Solve this equation for the case where

$$f = x^4 y''^2 + 4y^2 y',$$

in the interval $[0, 1]$ and

$$x^2 y(x) \to 0, \quad xy(x) \to 1$$

as $x \to 0$, whilst

$$y(1) = 2, \quad y'(1) = 0.$$

1/I/9D **Quantum Mechanics**

Consider a quantum mechanical particle of mass m moving in one dimension, in a potential well

$$V(x) = \begin{cases} \infty, & x < 0, \\ 0, & 0 < x < a, \\ V_0, & x > a. \end{cases}$$

Sketch the ground state energy eigenfunction $\chi(x)$ and show that its energy is $E = \frac{\hbar^2 k^2}{2m}$, where k satisfies

$$\tan ka = -\frac{k}{\sqrt{\frac{2mV_0}{\hbar^2} - k^2}}.$$

[*Hint: You may assume that* $\chi(0) = 0$.]

1/II/18D **Quantum Mechanics**

A quantum mechanical particle of mass M moves in one dimension in the presence of a negative delta function potential

$$V = -\frac{\hbar^2}{2M\Delta}\delta(x),$$

where Δ is a parameter with dimensions of length.

(a) Write down the time-independent Schrödinger equation for energy eigenstates $\chi(x)$, with energy E. By integrating this equation across $x = 0$, show that the gradient of the wavefunction jumps across $x = 0$ according to

$$\lim_{\epsilon \to 0} \left(\frac{d\chi}{dx}(\epsilon) - \frac{d\chi}{dx}(-\epsilon) \right) \;=\; -\frac{1}{\Delta}\chi(0).$$

[*You may assume that* χ *is continuous across* $x = 0$.]

(b) Show that there exists a negative energy solution and calculate its energy.

(c) Consider a double delta function potential

$$V(x) = -\frac{\hbar^2}{2M\Delta}[\delta(x+a) + \delta(x-a)].$$

For sufficiently small Δ, this potential yields a negative energy solution of odd parity, i.e. $\chi(-x) = -\chi(x)$. Show that its energy is given by

$$E = -\frac{\hbar^2}{2M}\lambda^2, \quad \text{where} \quad \tanh \lambda a = \frac{\lambda\Delta}{1 - \lambda\Delta}.$$

[*You may again assume* χ *is continuous across* $x = \pm a$.]

2/I/9D **Quantum Mechanics**

From the expressions

$$L_x = yP_z - zP_y, \quad L_y = zP_x - xP_z, \quad L_z = xP_y - yP_x,$$

show that

$$(x + iy)z$$

is an eigenfunction of $\mathbf{L}^2$ and L_z, and compute the corresponding eigenvalues.

2/II/18D **Quantum Mechanics**

Consider a quantum mechanical particle moving in an upside-down harmonic oscillator potential. Its wavefunction $\Psi(x,t)$ evolves according to the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\Psi}{\partial x^2} - \frac{1}{2}x^2\Psi. \tag{1}$$

(a) Verify that

$$\Psi(x,t) = A(t)e^{-B(t)x^2} \tag{2}$$

is a solution of equation (1), provided that

$$\frac{dA}{dt} = -i\hbar\, AB,$$

and

$$\frac{dB}{dt} = -\frac{i}{2\hbar} - 2i\hbar B^2. \tag{3}$$

(b) Verify that $B = \frac{1}{2\hbar}\tan(\phi - it)$ provides a solution to (3), where ϕ is an arbitrary real constant.

(c) The expectation value of an operator $\mathcal{O}$ at time t is

$$\langle\mathcal{O}\rangle(t) \equiv \int_{-\infty}^{\infty} dx\Psi^*(x,t)\mathcal{O}\Psi(x,t),$$

where $\Psi(x,t)$ is the normalised wave function. Show that for $\Psi(x,t)$ given by (2),

$$\langle x^2\rangle = \frac{1}{4Re(B)}, \quad \langle p^2\rangle = 4\hbar^2|B|^2\langle x^2\rangle.$$

Hence show that as $t\to\infty$,

$$\langle x^2\rangle \approx \langle p^2\rangle \approx \frac{\hbar}{4\sin 2\phi}e^{2t}.$$

[*Hint: You may use*

$$\frac{\int_{-\infty}^{\infty} dx\, e^{-Cx^2}x^2}{\int_{-\infty}^{\infty} dx\, e^{-Cx^2}} = \frac{1}{2C}.]$$

3/II/20D **Quantum Mechanics**

A quantum mechanical system has two states χ_0 and χ_1, which are normalised energy eigenstates of a Hamiltonian H_3, with

$$H_3\chi_0 = -\chi_0, \quad H_3\chi_1 = +\chi_1.$$

A general time-dependent state may be written

$$\Psi(t) = a_0(t)\chi_0 + a_1(t)\chi_1, \tag{1}$$

where $a_0(t)$ and $a_1(t)$ are complex numbers obeying $|a_0(t)|^2 + |a_1(t)|^2 = 1$.

(a) Write down the time-dependent Schrödinger equation for $\Psi(t)$, and show that if the Hamiltonian is H_3, then

$$i\hbar\frac{da_0}{dt} = -a_0, \qquad i\hbar\frac{da_1}{dt} = +a_1.$$

For the general state given in equation (1) above, write down the probability to observe the system, at time t, in a state $\alpha\chi_0 + \beta\chi_1$, properly normalised so that $|\alpha|^2 + |\beta|^2 = 1$.

(b) Now consider starting the system in the state χ_0 at time $t = 0$, and evolving it with a different Hamiltonian H_1, which acts on the states χ_0 and χ_1 as follows:

$$H_1\chi_0 = \chi_1, \qquad H_1\chi_1 = \chi_0.$$

By solving the time-dependent Schrödinger equation for the Hamiltonian H_1, find $a_0(t)$ and $a_1(t)$ in this case. Hence determine the shortest time $T > 0$ such that $\Psi(T)$ is an eigenstate of H_3 with eigenvalue $+1$.

(c) Now consider taking the state $\Psi(T)$ from part (b), and evolving it for further length of time T, with Hamiltonian H_2, which acts on the states χ_0 and χ_1 as follows:

$$H_2\chi_0 = -i\chi_1, \qquad H_2\chi_1 = i\chi_0.$$

What is the final state of the system? Is this state observationally distinguishable from the original state χ_0?

3/I/10D **Special Relativity**

Write down the formulae for a Lorentz transformation with velocity v taking one set of co-ordinates (t, x) to another (t', x').

Imagine you observe a train travelling past Cambridge station at a relativistic speed u. Someone standing still on the train throws a ball in the direction the train is moving, with speed v. How fast do you observe the ball to be moving? Justify your answer.

4/I/9D **Special Relativity**

A particle with mass M is observed to be at rest. It decays into a particle of mass $m < M$, and a massless particle. Calculate the energies and momenta of both final particles.

4/II/18D **Special Relativity**

A javelin of length 2m is thrown horizontally and lengthwise into a shed of length 1.5m at a speed of $0.8c$, where c is the speed of light.

(a) What is the length of the javelin in the rest frame of the shed?

(b) What is the length of the shed in the rest frame of the javelin?

(c) Draw a space-time diagram in the rest frame coordinates (ct, x) of the shed, showing the world lines of both ends of the javelin, and of the front and back of the shed. Draw a second space-time diagram in the rest frame coordinates (ct', x') of the javelin, again showing the world lines of both ends of the javelin and of the front and back of the shed.

(d) Clearly mark the space-time events corresponding to (A) the trailing end of the javelin entering the shed, and (B) the leading end of the javelin hitting the back of the shed. Give the corresponding (ct, x) and (ct', x') coordinates for both (A) and (B). Are these two events space-like, null or time-like separated? How does the javelin fit inside the shed, even though it is initially longer than the shed in its own rest frame?

1/I/6C **Fluid Dynamics**

A fluid flow has velocity given in Cartesian co-ordinates as $\mathbf{u} = (kty, 0, 0)$ where k is a constant and t is time. Show that the flow is incompressible. Find a stream function and determine an equation for the streamlines at time t.

At $t = 0$ the points along the straight line segment $x = 0,\ 0 \leqslant y \leqslant a,\ z = 0$ are marked with dye. Show that at any later time the marked points continue to form a segment of a straight line. Determine the length of this line segment at time t and the angle that it makes with the x-axis.

1/II/15C **Fluid Dynamics**

State the unsteady form of Bernoulli's theorem.

A spherical bubble having radius R_0 at time $t = 0$ is located with its centre at the origin in unbounded fluid. The fluid is inviscid, has constant density ρ and is everywhere at rest at $t = 0$. The pressure at large distances from the bubble has the constant value p_∞, and the pressure inside the bubble has the constant value $p_\infty - \triangle p$. In consequence the bubble starts to collapse so that its radius at time t is $R(t)$. Find the velocity everywhere in the fluid in terms of $R(t)$ at time t and, assuming that surface tension is negligible, show that R satisfies the equation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = -\frac{\triangle p}{\rho} \ .$$

Find the total kinetic energy of the fluid in terms of $R(t)$ at time t. Hence or otherwise obtain a first integral of the above equation.

3/I/8C **Fluid Dynamics**

State and prove Kelvin's circulation theorem.

Consider a planar flow in the unbounded region outside a cylinder for which the vorticity vanishes everywhere at time $t = 0$. What may be deduced about the circulation around closed loops in the fluid at time t:

(i) that do not enclose the cylinder;

(ii) that enclose the cylinder?

Give a brief justification for your answer in each case.

3/II/18C **Fluid Dynamics**

Use Euler's equation to derive Bernoulli's theorem for the steady flow of an inviscid fluid of uniform density ρ in the absence of body forces.

Such a fluid flows steadily through a long cylindrical elastic tube having circular cross-section. The variable z measures distance downstream along the axis of the tube. The tube wall has thickness $h(z)$, so that if the external radius of the tube is $r(z)$, its internal radius is $r(z) - h(z)$, where $h(z) \geqslant 0$ is a given slowly-varying function that tends to zero as $z \to \pm\infty$. The elastic tube wall exerts a pressure $p(z)$ on the fluid given as

$$p(z) = p_0 + k[r(z) - R],$$

where p_0, k and R are positive constants. Far upstream, r has the constant value R, the fluid pressure has the constant value p_0, and the fluid velocity u has the constant value V. Assume that gravity is negligible and that $h(z)$ varies sufficiently slowly that the velocity may be taken as uniform across the tube at each value of z. Use mass conservation and Bernoulli's theorem to show that $u(z)$ satisfies

$$\frac{h}{R} = 1 - \left(\frac{V}{u}\right)^{1/2} + \frac{1}{4}\lambda\left[1 - \left(\frac{u}{V}\right)^2\right], \quad \text{where} \quad \lambda = \frac{2\rho V^2}{kR} \ .$$

Sketch a graph of h/R against u/V. Show that if $h(z)$ exceeds a critical value $h_c(\lambda)$, no such flow is possible and find $h_c(\lambda)$.

Show that if $h < h_c(\lambda)$ everywhere, then for given h the equation has two positive solutions for u. Explain how the given value of λ determines which solution should be chosen.

4/I/7C **Fluid Dynamics**

If $\mathbf{u}$ is given in Cartesian co-ordinates as $\mathbf{u} = (-\Omega y, \Omega x, 0)$, with Ω a constant, verify that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla(-\tfrac{1}{2}\mathbf{u}^2) \ .$$

When incompressible fluid is placed in a stationary cylindrical container of radius a with its axis vertical, the depth of the fluid is h. Assuming that the free surface does not reach the bottom of the container, use cylindrical polar co-ordinates to find the equation of the free surface when the fluid and the container rotate steadily about this axis with angular velocity Ω.

Deduce the angular velocity at which the free surface first touches the bottom of the container.

4/II/16C **Fluid Dynamics**

Use Euler's equation to show that in a planar flow of an inviscid fluid the vorticity $\boldsymbol{\omega}$ satisfies

$$\frac{D\boldsymbol{\omega}}{Dt} = 0 \ .$$

Write down the velocity field associated with a point vortex of strength κ in unbounded fluid.

Consider now the flow generated in unbounded fluid by two point vortices of strengths κ_1 and κ_2 at $\mathbf{x}_1(t) = (x_1, y_1)$ and $\mathbf{x}_2(t) = (x_2, y_2)$, respectively. Show that in the subsequent motion the quantity

$$\mathbf{q} = \kappa_1 \mathbf{x}_1 + \kappa_2 \mathbf{x}_2$$

remains constant. Show also that the separation of the vortices, $|\mathbf{x}_2 - \mathbf{x}_1|$, remains constant.

Suppose finally that $\kappa_1 = \kappa_2$ and that the vortices are placed at time $t = 0$ at positions $(a, 0)$ and $(-a, 0)$. What are the positions of the vortices at time t?

2/I/5B **Numerical Analysis**

Applying the Gram–Schmidt orthogonalization, compute a "skinny" QR-factorization of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 6 \\ 1 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix},$$

i.e. find a 4×3 matrix Q with orthonormal columns and an upper triangular 3×3 matrix R such that $A = QR$.

2/II/14B **Numerical Analysis**

Let $f \in C[a,b]$ and let $n+1$ distinct points $x_0, \ldots, x_n \in [a,b]$ be given.

(a) Define the divided difference $f[x_0, \ldots, x_n]$ of order n in terms of interpolating polynomials. Prove that it is a symmetric function of the variables x_i, $i = 0, \ldots, n$.

(b) Prove the recurrence relation

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

(c) Hence or otherwise deduce that, for any $i \neq j$, we have

$$f[x_0, \ldots, x_n] = \frac{f[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] - f[x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n]}{x_j - x_i}.$$

(d) From the formulas above, show that, for any $i = 1, \ldots, n-1$,

$$f[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] = \gamma f[x_0, \ldots, x_{n-1}] + (1-\gamma) f[x_1, \ldots, x_n],$$

where $\gamma = \dfrac{x_i - x_0}{x_n - x_0}$.

3/I/6B **Numerical Analysis**

For numerical integration, a quadrature formula

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} a_i f(x_i)$$

is applied which is exact on $\mathcal{P}_n$, i.e., for all polynomials of degree n.

Prove that such a formula is exact for all $f \in \mathcal{P}_{2n+1}$ if and only if x_i, $i = 0, \ldots, n$, are the zeros of an orthogonal polynomial $p_{n+1} \in \mathcal{P}_{n+1}$ which satisfies $\int_a^b p_{n+1}(x) r(x)\,dx = 0$ for all $r \in \mathcal{P}_n$. [*You may assume that p_{n+1} has $(n+1)$ distinct zeros.*]

3/II/16B **Numerical Analysis**

(a) Consider a system of linear equations $Ax = b$ with a non-singular square $n \times n$ matrix A. To determine its solution $x = x^*$ we apply the iterative method

$$x^{k+1} = Hx^k + v.$$

Here $v \in \mathbb{R}^n$, while the matrix $H \in \mathbb{R}^{n \times n}$ is such that $x^* = Hx^* + v$ implies $Ax^* = b$. The initial vector $x^0 \in \mathbb{R}^n$ is arbitrary. Prove that, if the matrix H possesses n linearly independent eigenvectors $w_1, \ldots, w_n$ whose corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ satisfy $\max_i |\lambda_i| < 1$, then the method converges for any choice of x^0, i.e. $x^k \to x^*$ as $k \to \infty$.

(b) Describe the Jacobi iteration method for solving $Ax = b$. Show directly from the definition of the method that, if the matrix A is strictly diagonally dominant by rows, i.e.

$$|a_{ii}|^{-1} \sum_{j=1, j \neq i}^{n} |a_{ij}| \leq \gamma < 1, \quad i = 1, \ldots, n,$$

then the method converges.

1/I/3H **Statistics**

State the factorization criterion for sufficient statistics and give its proof in the discrete case.

Let $X_1, \ldots, X_n$ form a random sample from a Poisson distribution for which the value of the mean θ is unknown. Find a one-dimensional sufficient statistic for θ.

1/II/12H **Statistics**

Suppose we ask 50 men and 150 women whether they are early risers, late risers, or risers with no preference. The data are given in the following table.

	Early risers	*Late risers*	*No preference*	*Totals*
Men	17	22	11	50
Women	43	78	29	150
Totals	60	100	40	200

Derive carefully a (generalized) likelihood ratio test of independence of classification. What is the result of applying this test at the 0.01 level?

[*Distribution* χ_1^2 χ_2^2 χ_3^2 χ_5^2 χ_6^2
99%*percentile* *6.63* *9.21* *11.34* *15.09* *16.81*]

2/I/3H **Statistics**

Explain what is meant by a uniformly most powerful test, its power function and size.

Let $Y_1, \ldots, Y_n$ be independent identically distributed random variables with common density $\rho e^{-\rho y}$, $y \geq 0$. Obtain the uniformly most powerful test of $\rho = \rho_0$ against alternatives $\rho < \rho_0$ and determine the power function of the test.

2/II/12H Statistics

For ten steel ingots from a production process the following measures of hardness were obtained:

73.2, 74.3, 75.4, 73.8, 74.4, 76.7, 76.1, 73.0, 74.6, 74.1.

On the assumption that the variation is well described by a normal density function obtain an estimate of the process mean.

The manufacturer claims that he is supplying steel with mean hardness 75. Derive carefully a (generalized) likelihood ratio test of this claim. Knowing that for the data above

$$S_{XX} = \sum_{j=1}^{n} (X_i - \bar{X})^2 = 12.824,$$

what is the result of the test at the 5% significance level?

[

Distribution	t_9	t_{10}
95% percentile	*1.83*	*1.81*
97.5% percentile	*2.26*	*2.23*

]

4/I/3H Statistics

From each of 100 concrete mixes six sample blocks were taken and subjected to strength tests, the number out of the six blocks failing the test being recorded in the following table:

No. x failing strength tests	0	1	2	3	4	5	6
No. of mixes with x failures	53	32	12	2	1	0	0

On the assumption that the probability of failure is the same for each block, obtain an unbiased estimate of this probability and explain how to find a 95% confidence interval for it.

4/II/12H Statistics

Explain what is meant by a prior distribution, a posterior distribution, and a Bayes estimator. Relate the Bayes estimator to the posterior distribution for both quadratic and absolute error loss functions.

Suppose $X_1, \ldots, X_n$ are independent identically distributed random variables from a distribution uniform on $(\theta - 1, \theta + 1)$, and that the prior for θ is uniform on $(20, 50)$.

Calculate the posterior distribution for θ, given $\mathbf{x} = (x_1, \ldots, x_n)$, and find the point estimate for θ under both quadratic and absolute error loss function.

3/I/5H Optimization

Consider a two-person zero-sum game with a payoff matrix

$$\begin{pmatrix} 3 & b \\ 5 & 2 \end{pmatrix},$$

where $0 < b < \infty$. Here, the (i, j) entry of the matrix indicates the payoff to player one if he chooses move i and player two move j. Suppose player one chooses moves 1 and 2 with probabilities p and $1 - p$, $0 \leq p \leq 1$. Write down the maximization problem for the optimal strategy and solve it for each value of b.

3/II/15H Optimization

Consider the following linear programming problem

$$\begin{array}{ll} \text{maximise} & -2x_1 + 3x_2 \\ \text{subject to} & x_1 - x_2 \geq 1, \\ & 4x_1 - x_2 \geq 10, \\ & x_2 \leq 6, \\ & x_i \geq 0,\ i = 1, 2. \end{array} \tag{1}$$

Write down the Phase One problem for (1) and solve it.

By using the solution of the Phase One problem as an initial basic feasible solution for the Phase Two simplex algorithm, solve (1), i.e., find the optimal tableau and read the optimal solution (x_1, x_2) and optimal value from it.

4/I/5H Optimization

State and prove the max flow/min cut theorem. In your answer you should define clearly the following terms: flow, maximal flow, cut, capacity.

4/II/14H **Optimization**

A gambler at a horse race has an amount $b > 0$ to bet. The gambler assesses p_i, the probability that horse i will win, and knows that $s_i \geq 0$ has been bet on horse i by others, for $i = 1, 2, \ldots, n$. The total amount bet on the race is shared out in proportion to the bets on the winning horse, and so the gambler's optimal strategy is to choose $(x_1, x_2, \ldots, x_n)$ so that it maximizes

$$\sum_{i=1}^{n} \frac{p_i x_i}{s_i + x_i} \qquad \text{subject to } \sum_{i=1}^{n} x_i = b, \quad x_1, \ldots, x_n \geq 0, \tag{1}$$

where x_i is the amount the gambler bets on horse i. Show that the optimal solution to (1) also solves the following problem:

$$\text{minimize } \sum_{i=1}^{n} \frac{p_i s_i}{s_i + x_i} \qquad \text{subject to } \sum_{i=1}^{n} x_i = b, \quad x_1, \ldots, x_n \geq 0.$$

Assume that $p_1/s_1 \geq p_2/s_2 \geq \ldots \geq p_n/s_n$. Applying the Lagrangian sufficiency theorem, prove that the optimal solution to (1) satisfies

$$\frac{p_1 s_1}{(s_1 + x_1)^2} = \ldots = \frac{p_k s_k}{(s_k + x_k)^2}, \quad x_{k+1} = \ldots = x_n = 0,$$

with maximal possible $k \in \{1, 2, \ldots, n\}$.

[*You may use the fact that for all $\lambda < 0$, the minimum of the function $x \mapsto \frac{ps}{s+x} - \lambda x$ on the non-negative axis $0 \leq x < \infty$ is attained at*

$$x(\lambda) = \left(\sqrt{\frac{ps}{-\lambda}} - s\right)^{+} \equiv \max\left(\sqrt{\frac{ps}{-\lambda}} - s, 0\right).]$$

Deduce that if b is small enough, the gambler's optimal strategy is to bet on the horses for which the ratio p_i/s_i is maximal. What is his expected gain in this case?

1/I/8F Quadratic Mathematics

Define the *rank* and *signature* of a symmetric bilinear form ϕ on a finite-dimensional real vector space. (If your definitions involve a matrix representation of ϕ, you should explain why they are independent of the choice of representing matrix.)

Let V be the space of all $n \times n$ real matrices (where $n \geqslant 2$), and let ϕ be the bilinear form on V defined by

$$\phi(A, B) = \operatorname{tr} AB - \operatorname{tr} A \operatorname{tr} B \ .$$

Find the rank and signature of ϕ.

[*Hint: You may find it helpful to consider the subspace of symmetric matrices having trace zero, and a suitable complement for this subspace.*]

1/II/17F Quadratic Mathematics

Let A and B be $n \times n$ real symmetric matrices, such that the quadratic form $\mathbf{x}^T A \mathbf{x}$ is positive definite. Show that it is possible to find an invertible matrix P such that $P^T AP = I$ and $P^T BP$ is diagonal. Show also that the diagonal entries of the matrix $P^T BP$ may be calculated directly from A and B, without finding the matrix P. If

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$

find the diagonal entries of $P^T BP$.

2/I/8F Quadratic Mathematics

Explain what is meant by a *sesquilinear form* on a complex vector space V. If ϕ and ψ are two such forms, and $\phi(v, v) = \psi(v, v)$ for all $v \in V$, prove that $\phi(v, w) = \psi(v, w)$ for all $v, w \in V$. Deduce that if $\alpha \colon V \to V$ is a linear map satisfying $\phi(\alpha(v), \alpha(v)) = \phi(v, v)$ for all $v \in V$, then $\phi(\alpha(v), \alpha(w)) = \phi(v, w)$ for all $v, w \in V$.

2/II/17F Quadratic Mathematics

Define the *adjoint* α^* of an endomorphism α of a complex inner-product space V. Show that if W is a subspace of V, then $\alpha(W) \subseteq W$ if and only if $\alpha^*(W^\perp) \subseteq W^\perp$.

An endomorphism of a complex inner-product space is said to be *normal* if it commutes with its adjoint. Prove the following facts about a normal endomorphism α of a finite-dimensional space V.

(i) α and α^* have the same kernel.

(ii) α and α^* have the same eigenvectors, with complex conjugate eigenvalues.

(iii) If $E_\lambda = \{x \in V : \alpha(x) = \lambda x\}$, then $\alpha(E_\lambda^\perp) \subseteq E_\lambda^\perp$.

(iv) There is an orthonormal basis of V consisting of eigenvectors of α.

Deduce that an endomorphism α is normal if and only if it can be written as a product $\beta\gamma$, where β is Hermitian, γ is unitary and β and γ commute with each other. [*Hint: Given α, define β and γ in terms of their effect on the basis constructed in (iv).*]

3/I/9F Quadratic Mathematics

Explain what is meant by a *quadratic residue* modulo an odd prime p, and show that a is a quadratic residue modulo p if and only if $a^{\frac{1}{2}(p-1)} \equiv 1 \pmod{p}$. Hence characterize the odd primes p for which -1 is a quadratic residue.

State the law of quadratic reciprocity, and use it to determine whether 73 is a quadratic residue (mod 127).

3/II/19F Quadratic Mathematics

Explain what is meant by saying that a positive definite integral quadratic form $f(x,y) = ax^2+bxy+cy^2$ is *reduced*, and show that every positive definite form is equivalent to a reduced form.

State a criterion for a prime number p to be representable by some form of discriminant d, and deduce that p is representable by a form of discriminant -32 if and only if $p \equiv 1, 2$ or $3 \pmod 8$. Find the reduced forms of discriminant -32, and hence or otherwise show that a prime p is representable by the form $3x^2 + 2xy + 3y^2$ if and only if $p \equiv 3 \pmod 8$.

[*Standard results on when -1 and 2 are squares* (mod p) *may be assumed.*]

2/I/4G Further Analysis

Let the function $f = u + iv$ be analytic in the complex plane $\mathbb{C}$ with u,v real-valued. Prove that, if uv is bounded above everywhere on $\mathbb{C}$, then f is constant.

2/II/13G Further Analysis

(a) Given a topology $\mathcal{T}$ on X, a collection $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis* for $\mathcal{T}$ if every non-empty set in $\mathcal{T}$ is a union of sets in $\mathcal{B}$. Prove that a collection $\mathcal{B}$ is a basis for some topology if it satisfies:

(i) the union of all sets in $\mathcal{B}$ is X;

(ii) if $x \in B_1 \cap B_2$ for two sets B_1 and B_2 in $\mathcal{B}$, then there is a set $B \in \mathcal{B}$ with $x \in B \subset B_1 \cap B_2$.

(b) On $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ consider the dictionary order given by

$$(a_1, b_1) < (a_2, b_2)$$

if $a_1 < a_2$ or if $a_1 = a_2$ and $b_1 < b_2$. Given points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ let

$$\langle \mathbf{x}, \mathbf{y} \rangle = \{\mathbf{z} \in \mathbb{R}^2 : \ \mathbf{x} < \mathbf{z} < \mathbf{y}\}.$$

Show that the sets $\langle \mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^2$ form a basis of a topology.

(c) Show that this topology on $\mathbb{R}^2$ does not have a countable basis.

3/I/3G Further Analysis

Let $f : X \to Y$ be a continuous map between topological spaces. Let

$$G_f = \{(x, f(x)) : \ x \in X\}.$$

(a) Show that if Y is Hausdorff, then G_f is closed in $X \times Y$.

(b) Show that if X is compact, then G_f is also compact.

3/II/13G Further Analysis

(a) Let f and g be two analytic functions on a domain D and let $\gamma \subset D$ be a simple closed curve homotopic in D to a point. If $|g(z)| < |f(z)|$ for every z in γ, prove that γ encloses the same number of zeros of f as of $f+g$.

(b) Let g be an analytic function on the disk $|z| < 1+\epsilon$, for some $\epsilon > 0$. Suppose that g maps the closed unit disk into the open unit disk (both centred at 0). Prove that g has exactly one fixed point in the open unit disk.

(c) Prove that, if $|a| < 1$, then

$$z^m \left(\frac{z-a}{1-\bar{a}z} \right)^n - a$$

has $m+n$ zeros in $|z| < 1$.

4/I/4G Further Analysis

(a) Let X be a topological space and suppose $X = C \cup D$, where C and D are disjoint nonempty open subsets of X. Show that, if Y is a connected subset of X, then Y is entirely contained in either C or D.

(b) Let X be a topological space and let $\{A_n\}$ be a sequence of connected subsets of X such that $A_n \cap A_{n+1} \neq \emptyset$, for $n = 1, 2, 3, \ldots$. Show that $\bigcup_{n \geqslant 1} A_n$ is connected.

4/II/13G Further Analysis

A function f is said to be analytic at ∞ if there exists a real number $r > 0$ such that f is analytic for $|z| > r$ and $\lim_{z \to 0} f(1/z)$ is finite (i.e. $f(1/z)$ has a removable singularity at $z = 0$). f is said to have a pole at ∞ if $f(1/z)$ has a pole at $z = 0$. Suppose that f is a meromorphic function on the extended plane $\mathbb{C}_\infty$, that is, f is analytic at each point of $\mathbb{C}_\infty$ except for poles.

(a) Show that if f has a pole at $z = \infty$, then there exists $r > 0$ such that $f(z)$ has no poles for $r < |z| < \infty$.

(b) Show that the number of poles of f is finite.

(c) By considering the Laurent expansions around the poles show that f is in fact a rational function, i.e. of the form p/q, where p and q are polynomials.

(d) Deduce that the only bijective meromorphic maps of $\mathbb{C}_\infty$ onto itself are the Möbius maps.