UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday 7 June 2002 1.30 to 4.30

PAPER 4

Before you begin read these instructions carefully.

Each question in Section II carries twice the credit of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Answers must be tied up in separate bundles, marked ***A, B, . . . , H*** *according to the letter affixed to each question, and a blue cover sheet must be attached to each bundle.*

A green master cover sheet listing all the questions attempted must be completed.

It is essential that every cover sheet bear the candidate's examination number and desk number.

SECTION I

1E Analysis II

(a) Let (X, d) be a metric space containing the point x_0, and let

$$U = \{x \in X : d(x, x_0) < 1\}, \quad K = \{x \in X : d(x, x_0) \leqslant 1\}.$$

Is U necessarily the largest open subset of K? Is K necessarily the smallest closed set that contains U? Justify your answers.

(b) Let X be a normed space with norm $||\cdot||$, and let

$$U = \{x \in X : ||x|| < 1\}, \quad K = \{x \in X : ||x|| \leqslant 1\}.$$

Is U necessarily the largest open subset of K? Is K necessarily the smallest closed set that contains U? Justify your answers.

2A Methods

Use the method of Lagrange multipliers to find the largest volume of a rectangular parallelepiped that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

3H Statistics

From each of 100 concrete mixes six sample blocks were taken and subjected to strength tests, the number out of the six blocks failing the test being recorded in the following table:

No. x failing strength tests	0	1	2	3	4	5	6
No. of mixes with x failures	53	32	12	2	1	0	0

On the assumption that the probability of failure is the same for each block, obtain an unbiased estimate of this probability and explain how to find a 95% confidence interval for it.

4G Further Analysis

(a) Let X be a topological space and suppose $X = C \cup D$, where C and D are disjoint nonempty open subsets of X. Show that, if Y is a connected subset of X, then Y is entirely contained in either C or D.

(b) Let X be a topological space and let $\{A_n\}$ be a sequence of connected subsets of X such that $A_n \cap A_{n+1} \neq \emptyset$, for $n = 1, 2, 3, \ldots$. Show that $\bigcup_{n \geqslant 1} A_n$ is connected.

5H Optimization

State and prove the max flow/min cut theorem. In your answer you should define clearly the following terms: flow, maximal flow, cut, capacity.

6F Linear Mathematics

Define the *rank* and *nullity* of a linear map between finite-dimensional vector spaces. State the rank–nullity formula.

Let $\alpha \colon U \to V$ and $\beta \colon V \to W$ be linear maps. Prove that

$$\operatorname{rank}(\alpha) + \operatorname{rank}(\beta) - \dim V \leqslant \operatorname{rank}(\beta\alpha) \leqslant \min\{\operatorname{rank}(\alpha), \operatorname{rank}(\beta)\} \ .$$

7C Fluid Dynamics

If $\mathbf{u}$ is given in Cartesian co-ordinates as $\mathbf{u} = (-\Omega y, \Omega x, 0)$, with Ω a constant, verify that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla(-\tfrac{1}{2}\mathbf{u}^2) \ .$$

When incompressible fluid is placed in a stationary cylindrical container of radius a with its axis vertical, the depth of the fluid is h. Assuming that the free surface does not reach the bottom of the container, use cylindrical polar co-ordinates to find the equation of the free surface when the fluid and the container rotate steadily about this axis with angular velocity Ω.

Deduce the angular velocity at which the free surface first touches the bottom of the container.

8B Complex Methods

Let f be a function such that $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$. Prove that

$$\int_{-\infty}^{+\infty} f(x+k)\overline{f(x+l)}\, dx = 0 \quad \text{for all integers } k \text{ and } l \text{ with } k \neq l,$$

if and only if

$$\int_{-\infty}^{+\infty} |\widehat{f}(t)|^2 e^{-imt} dt = 0 \quad \text{for all integers } m \neq 0,$$

where $\widehat{f}$ is the Fourier transform of f.

9D Special Relativity

A particle with mass M is observed to be at rest. It decays into a particle of mass $m < M$, and a massless particle. Calculate the energies and momenta of both final particles.

SECTION II

10E Analysis II

(a) Let V be a finite-dimensional real vector space, and let $||\cdot||_1$ and $||\cdot||_2$ be two norms on V. Show that a function $f : V \to \mathbb{R}$ is differentiable at a point a in V with respect to $||\cdot||_1$ if and only if it is differentiable at a with respect to $||\cdot||_2$, and that if this is so then the derivative $f'(a)$ of f is independent of the norm used. [*You may assume that all norms on a finite-dimensional vector space are equivalent.*]

(b) Let V_1, V_2 and V_3 be finite-dimensional normed real vector spaces with V_j having norm $||\cdot||_j$, $j = 1, 2, 3$, and let $f : V_1 \times V_2 \to V_3$ be a continuous bilinear mapping. Show that f is differentiable at any point (a, b) in $V_1 \times V_2$, and that $f'(a, b)(h, k) = f(h, b) + f(a, k)$. [*You may assume that* $\left(||u||_1^2 + ||v||_2^2\right)^{1/2}$ *is a norm on* $V_1 \times V_2$*, and that* $\{(x, y) \in V_1 \times V_2 : ||x||_1 = 1, ||y||_2 = 1\}$ *is compact.*]

11A Methods

A function $y(x)$ is chosen to make the integral

$$I = \int_a^b f\left(x, y, y', y''\right) dx$$

stationary, subject to given values of $y(a), y'(a), y(b)$ and $y'(b)$. Derive an analogue of the Euler–Lagrange equation for $y(x)$.

Solve this equation for the case where

$$f = x^4 y''^2 + 4y^2 y',$$

in the interval $[0, 1]$ and

$$x^2 y(x) \to 0, \quad xy(x) \to 1$$

as $x \to 0$, whilst

$$y(1) = 2, \quad y'(1) = 0.$$

12H Statistics

Explain what is meant by a prior distribution, a posterior distribution, and a Bayes estimator. Relate the Bayes estimator to the posterior distribution for both quadratic and absolute error loss functions.

Suppose $X_1, \ldots, X_n$ are independent identically distributed random variables from a distribution uniform on $(\theta - 1, \theta + 1)$, and that the prior for θ is uniform on $(20, 50)$.

Calculate the posterior distribution for θ, given $\mathbf{x} = (x_1, \ldots, x_n)$, and find the point estimate for θ under both quadratic and absolute error loss function.

13G Further Analysis

A function f is said to be analytic at ∞ if there exists a real number $r > 0$ such that f is analytic for $|z| > r$ and $\lim_{z \to 0} f(1/z)$ is finite (i.e. $f(1/z)$ has a removable singularity at $z = 0$). f is said to have a pole at ∞ if $f(1/z)$ has a pole at $z = 0$. Suppose that f is a meromorphic function on the extended plane $\mathbb{C}_\infty$, that is, f is analytic at each point of $\mathbb{C}_\infty$ except for poles.

(a) Show that if f has a pole at $z = \infty$, then there exists $r > 0$ such that $f(z)$ has no poles for $r < |z| < \infty$.

(b) Show that the number of poles of f is finite.

(c) By considering the Laurent expansions around the poles show that f is in fact a rational function, i.e. of the form p/q, where p and q are polynomials.

(d) Deduce that the only bijective meromorphic maps of $\mathbb{C}_\infty$ onto itself are the Möbius maps.

14H Optimization

A gambler at a horse race has an amount $b > 0$ to bet. The gambler assesses p_i, the probability that horse i will win, and knows that $s_i \geq 0$ has been bet on horse i by others, for $i = 1, 2, \ldots, n$. The total amount bet on the race is shared out in proportion to the bets on the winning horse, and so the gambler's optimal strategy is to choose $(x_1, x_2, \ldots, x_n)$ so that it maximizes

$$\sum_{i=1}^{n} \frac{p_i x_i}{s_i + x_i} \qquad \text{subject to } \sum_{i=1}^{n} x_i = b, \quad x_1, \ldots, x_n \geq 0, \qquad (1)$$

where x_i is the amount the gambler bets on horse i. Show that the optimal solution to (1) also solves the following problem:

$$\text{minimize } \sum_{i=1}^{n} \frac{p_i s_i}{s_i + x_i} \qquad \text{subject to } \sum_{i=1}^{n} x_i = b, \quad x_1, \ldots, x_n \geq 0.$$

Assume that $p_1/s_1 \geq p_2/s_2 \geq \ldots \geq p_n/s_n$. Applying the Lagrangian sufficiency theorem, prove that the optimal solution to (1) satisfies

$$\frac{p_1 s_1}{(s_1 + x_1)^2} = \ldots = \frac{p_k s_k}{(s_k + x_k)^2}, \quad x_{k+1} = \ldots = x_n = 0,$$

with maximal possible $k \in \{1, 2, \ldots, n\}$.

[*You may use the fact that for all* $\lambda < 0$, *the minimum of the function* $x \mapsto \frac{ps}{s+x} - \lambda x$ *on the non-negative axis* $0 \leq x < \infty$ *is attained at*

$$x(\lambda) = \left(\sqrt{\frac{ps}{-\lambda}} - s\right)^+ \equiv \max\left(\sqrt{\frac{ps}{-\lambda}} - s, 0\right).]$$

Deduce that if b is small enough, the gambler's optimal strategy is to bet on the horses for which the ratio p_i/s_i is maximal. What is his expected gain in this case?

15F Linear Mathematics

Define the *dual space* V^* of a finite-dimensional real vector space V, and explain what is meant by the basis of V^* dual to a given basis of V. Explain also what is meant by the statement that the second dual V^{**} is naturally isomorphic to V.

Let V_n denote the space of real polynomials of degree at most n. Show that, for any real number x, the function e_x mapping p to $p(x)$ is an element of V_n^*. Show also that, if $x_1, x_2, \ldots, x_{n+1}$ are distinct real numbers, then $\{e_{x_1}, e_{x_2}, \ldots, e_{x_{n+1}}\}$ is a basis of V_n^*, and find the basis of V_n dual to it.

Deduce that, for any $(n+1)$ distinct points $x_1, \ldots, x_{n+1}$ of the interval $[-1, 1]$, there exist scalars $\lambda_1, \ldots, \lambda_{n+1}$ such that

$$\int_{-1}^{1} p(t)\, dt = \sum_{i=1}^{n+1} \lambda_i p(x_i)$$

for all $p \in V_n$. For $n = 4$ and $(x_1, x_2, x_3, x_4, x_5) = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$, find the corresponding scalars λ_i.

16C Fluid Dynamics

Use Euler's equation to show that in a planar flow of an inviscid fluid the vorticity $\boldsymbol{\omega}$ satisfies

$$\frac{D\boldsymbol{\omega}}{Dt} = 0\ .$$

Write down the velocity field associated with a point vortex of strength κ in unbounded fluid.

Consider now the flow generated in unbounded fluid by two point vortices of strengths κ_1 and κ_2 at $\mathbf{x}_1(t) = (x_1, y_1)$ and $\mathbf{x}_2(t) = (x_2, y_2)$, respectively. Show that in the subsequent motion the quantity

$$\mathbf{q} = \kappa_1 \mathbf{x}_1 + \kappa_2 \mathbf{x}_2$$

remains constant. Show also that the separation of the vortices, $|\mathbf{x}_2 - \mathbf{x}_1|$, remains constant.

Suppose finally that $\kappa_1 = \kappa_2$ and that the vortices are placed at time $t = 0$ at positions $(a, 0)$ and $(-a, 0)$. What are the positions of the vortices at time t?

17B Complex Methods

(a) Using the inequality $\sin\theta \geq 2\theta/\pi$ for $0 \leq \theta \leq \frac{\pi}{2}$, show that, if f is continuous for large $|z|$, and if $f(z) \to 0$ as $z \to \infty$, then

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)e^{i\lambda z} dz = 0 \quad \text{for} \quad \lambda > 0,$$

where $\Gamma_R = Re^{i\theta}$, $0 \leq \theta \leq \pi$.

(b) By integrating an appropriate function $f(z)$ along the contour formed by the semicircles Γ_R and Γ_r in the upper half-plane with the segments of the real axis $[-R, -r]$ and $[r, R]$, show that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

18D Special Relativity

A javelin of length 2m is thrown horizontally and lengthwise into a shed of length 1.5m at a speed of $0.8c$, where c is the speed of light.

(a) What is the length of the javelin in the rest frame of the shed?

(b) What is the length of the shed in the rest frame of the javelin?

(c) Draw a space-time diagram in the rest frame coordinates (ct, x) of the shed, showing the world lines of both ends of the javelin, and of the front and back of the shed. Draw a second space-time diagram in the rest frame coordinates (ct', x') of the javelin, again showing the world lines of both ends of the javelin and of the front and back of the shed.

(d) Clearly mark the space-time events corresponding to (A) the trailing end of the javelin entering the shed, and (B) the leading end of the javelin hitting the back of the shed. Give the corresponding (ct, x) and (ct', x') coordinates for both (A) and (B). Are these two events space-like, null or time-like separated? How does the javelin fit inside the shed, even though it is initially longer than the shed in its own rest frame?

END OF PAPER