UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 4 June, 2012 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Numbers and Sets

(i) Find integers x and y such that $18x + 23y = 101$.

(ii) Find an integer x such that $x \equiv 3 \pmod{18}$ and $x \equiv 2 \pmod{23}$.

2D Numbers and Sets

What is an *equivalence relation* on a set X? If R is an equivalence relation on X, what is an *equivalence class* of R? Prove that the equivalence classes of R form a partition of X.

Let R and S be equivalence relations on a set X. Which of the following are always equivalence relations? Give proofs or counterexamples as appropriate.

(i) The relation V on X given by xVy if both xRy and xSy.

(ii) The relation W on X given by xWy if xRy or xSy.

3B Dynamics and Relativity

Two particles of masses m_1 and m_2 have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Particle 2 exerts a force $\mathbf{F}_{12}(\mathbf{r})$ on particle 1 (where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$) and there are no external forces.

Prove that the centre of mass of the two-particle system will move at constant speed along a straight line.

Explain how the two-body problem of determining the motion of the system may be reduced to that of a single particle moving under the force $\mathbf{F}_{12}$.

Suppose now that $m_1 = m_2 = m$ and that

$$\mathbf{F}_{12} = -\frac{Gm^2}{r^3}\mathbf{r}$$

is gravitational attraction. Let C be a circle fixed in space. Is it possible (by suitable choice of initial conditions) for the two particles to be traversing C at the same constant angular speed? Give a brief reason for your answer.

4B Dynamics and Relativity

Let S and S' be inertial frames in 2-dimensional spacetime with coordinate systems (t, x) and (t', x') respectively. Suppose that S' moves with positive velocity v relative to S and the spacetime origins of S and S' coincide. Write down the Lorentz transformation relating the coordinates of any event relative to the two frames.

Show that events which occur simultaneously in S are not generally seen to be simultaneous when viewed in S'.

In S two light sources A and B are at rest and placed a distance d apart. They simultaneously each emit a photon in the positive x direction. Show that in S' the photons are separated by a constant distance $d\sqrt{\dfrac{c+v}{c-v}}$.

SECTION II

5D Numbers and Sets

Let X be a set, and let f and g be functions from X to X. Which of the following are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) If fg is the identity map then gf is the identity map.

(ii) If $fg = g$ then f is the identity map.

(iii) If $fg = f$ then g is the identity map.

How (if at all) do your answers change if we are given that X is finite?

Determine which sets X have the following property: if f is a function from X to X such that for every $x \in X$ there exists a positive integer n with $f^n(x) = x$, then there exists a positive integer n such that f^n is the identity map. [Here f^n denotes the n-fold composition of f with itself.]

6D Numbers and Sets

State Fermat's Theorem and Wilson's Theorem.

For which prime numbers p does the equation $x^2 \equiv -1 \pmod{p}$ have a solution? Justify your answer.

For a prime number p, and an integer x that is not a multiple of p, the *order* of x (mod p) is the least positive integer d such that $x^d \equiv 1 \pmod{p}$. Show that if x has order d and also $x^k \equiv 1 \pmod{p}$ then d must divide k.

For a positive integer n, let $F_n = 2^{2^n} + 1$. If p is a prime factor of F_n, determine the order of 2 (mod p). Hence show that the F_n are pairwise coprime.

Show that if p is a prime of the form $4k + 3$ then p cannot be a factor of any F_n. Give, with justification, a prime p of the form $4k + 1$ such that p is not a factor of any F_n.

7D Numbers and Sets

Prove that each of the following numbers is irrational:

(i) $\sqrt{2} + \sqrt{3}$

(ii) e

(iii) The real root of the equation $x^3 + 4x - 7 = 0$

(iv) $\log_2 3$.

8D Numbers and Sets

Show that there is no injection from the power-set of $\mathbb{R}$ to $\mathbb{R}$. Show also that there *is* an injection from $\mathbb{R}^2$ to $\mathbb{R}$.

Let X be the set of all functions f from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x) = x$ for all but finitely many x. Determine whether or not there exists an injection from X to $\mathbb{R}$.

9B Dynamics and Relativity

Let (r, θ) be polar coordinates in the plane. A particle of mass m moves in the plane under an attractive force of $mf(r)$ towards the origin O. You may assume that the acceleration $\mathbf{a}$ is given by

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\theta}$$

where $\hat{\mathbf{r}}$ and $\hat{\theta}$ are the unit vectors in the directions of increasing r and θ respectively, and the dot denotes d/dt.

(a) Show that $l = r^2\dot{\theta}$ is a constant of the motion. Introducing $u = 1/r$ show that $\dot{r} = -l\dfrac{du}{d\theta}$ and derive the geometric orbit equation

$$l^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = f\left(\frac{1}{u}\right).$$

(b) Suppose now that

$$f(r) = \frac{3r + 9}{r^3}$$

and that initially the particle is at distance $r_0 = 1$ from O, moving with speed $v_0 = 4$ in a direction making angle $\pi/3$ with the radial vector pointing towards O.

Show that $l = 2\sqrt{3}$ and find u as a function of θ. Hence or otherwise show that the particle returns to its original position after one revolution about O and then flies off to infinity.

10B Dynamics and Relativity

For any frame S and vector $\mathbf{A}$, let $\left[\frac{d\mathbf{A}}{dt}\right]_S$ denote the derivative of $\mathbf{A}$ relative to S. A frame of reference S' rotates with constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame S and the two frames have a common origin O. [You may assume that for any vector $\mathbf{A}$, $\left[\frac{d\mathbf{A}}{dt}\right]_S = \left[\frac{d\mathbf{A}}{dt}\right]_{S'} + \boldsymbol{\omega} \times \mathbf{A}$.]

(a) If $\mathbf{r}(t)$ is the position vector of a point P from O, show that

$$\left[\frac{d^2\mathbf{r}}{dt^2}\right]_S = \left[\frac{d^2\mathbf{r}}{dt^2}\right]_{S'} + 2\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where $\mathbf{v}' = \left[\frac{d\mathbf{r}}{dt}\right]_{S'}$ is the velocity in S'.

Suppose now that $\mathbf{r}(t)$ is the position vector of a particle of mass m moving under a conservative force $\mathbf{F} = -\nabla\phi$ and a force $\mathbf{G}$ that is always orthogonal to the velocity $\mathbf{v}'$ in S'. Show that the quantity

$$E = \frac{1}{2}m\mathbf{v}'.\mathbf{v}' + \phi - \frac{m}{2}(\boldsymbol{\omega} \times \mathbf{r}).(\boldsymbol{\omega} \times \mathbf{r})$$

is a constant of the motion. [You may assume that $\nabla\left[(\boldsymbol{\omega} \times \mathbf{r}).(\boldsymbol{\omega} \times \mathbf{r})\right] = -2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$.]

(b) A bead slides on a frictionless circular hoop of radius a which is forced to rotate with constant angular speed ω about a vertical diameter. Let $\theta(t)$ denote the angle between the line from the centre of the hoop to the bead and the downward vertical. Using the results of (a), or otherwise, show that

$$\ddot{\theta} + \left(\frac{g}{a} - \omega^2 \cos\theta\right)\sin\theta = 0.$$

Deduce that if $\omega^2 > g/a$ there are two equilibrium positions $\theta = \theta_0$ off the axis of rotation, and show that these are stable equilibria.

11B Dynamics and Relativity

(a) State the parallel axis theorem for moments of inertia.

(b) A uniform circular disc D of radius a and total mass m can turn frictionlessly about a fixed horizontal axis that passes through a point A on its circumference and is perpendicular to its plane. Initially the disc hangs at rest (in constant gravity g) with its centre O being vertically below A. Suppose the disc is disturbed and executes free oscillations. Show that the period of small oscillations is $2\pi\sqrt{\dfrac{3a}{2g}}$.

(c) Suppose now that the disc is released from rest when the radius OA is vertical with O directly above A. Find the angular velocity and angular acceleration of O about A when the disc has turned through angle θ. Let $\mathbf{R}$ denote the reaction force at A on the disc. Find the acceleration of the centre of mass of the disc. Hence, or otherwise, show that the component of $\mathbf{R}$ parallel to OA is $mg(7\cos\theta - 4)/3$.

12B Dynamics and Relativity

(a) Define the 4-momentum $\mathbf{P}$ of a particle of rest mass m and 3-velocity $\mathbf{v}$, and the 4-momentum of a photon of frequency ν (having zero rest mass) moving in the direction of the unit vector $\mathbf{e}$.

Show that if $\mathbf{P}_1$ and $\mathbf{P}_2$ are timelike future-pointing 4-vectors then $\mathbf{P}_1.\mathbf{P}_2 \geqslant 0$ (where the dot denotes the Lorentz-invariant scalar product). Hence or otherwise show that the law of conservation of 4-momentum forbids a photon to spontaneously decay into an electron-positron pair. [Electrons and positrons have equal rest masses $m > 0$.]

(b) In the laboratory frame an electron travelling with velocity $\mathbf{u}$ collides with a positron at rest. They annihilate, producing two photons of frequencies ν_1 and ν_2 that move off at angles θ_1 and θ_2 to $\mathbf{u}$, in the directions of the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively. By considering 4-momenta in the laboratory frame, or otherwise, show that

$$\frac{1+\cos(\theta_1+\theta_2)}{\cos\theta_1+\cos\theta_2} = \sqrt{\frac{\gamma-1}{\gamma+1}}$$

where $\gamma = \left(1 - \dfrac{u^2}{c^2}\right)^{-1/2}$.

END OF PAPER