UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 2 June, 2011 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Vectors and Matrices

For $z, a \in \mathbb{C}$ define the *principal value* of $\log z$ and hence of z^a. Hence find all solutions to

$$\text{(i) } z^{\mathrm{i}} = 1$$

$$\text{(ii) } z^{\mathrm{i}} + \overline{z}^{\,\mathrm{i}} = 2\mathrm{i}\,,$$

and sketch the curve $|z^{\mathrm{i}+1}| = 1$.

2A Vectors and Matrices

The matrix

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ -1 & 1 \end{pmatrix}$$

represents a linear map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ with respect to the bases

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \quad , \qquad C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \; .$$

Find the matrix A' that represents Φ with respect to the bases

$$B' = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\} \quad , \qquad C' = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\} \; .$$

3F Analysis I

(a) State, without proof, the Bolzano–Weierstrass Theorem.

(b) Give an example of a sequence that does not have a convergent subsequence.

(c) Give an example of an unbounded sequence having a convergent subsequence.

(d) Let $a_n = 1 + (-1)^{\lfloor n/2 \rfloor}\big(1 + 1/n\big)$, where $\lfloor x \rfloor$ denotes the integer part of x. Find all values c such that the sequence $\{a_n\}$ has a subsequence converging to c. For each such value, provide a subsequence converging to it.

4D Analysis I

Find the radius of convergence of each of the following power series.

(i) $\displaystyle\sum_{n\geqslant 1} n^2 z^n$

(ii) $\displaystyle\sum_{n\geqslant 1} n^{n^{1/3}} z^n$

SECTION II

5C Vectors and Matrices

Explain why each of the equations

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{b} \tag{1}$$

$$\mathbf{x} \times \mathbf{c} = \mathbf{d} \tag{2}$$

describes a straight line, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are constant vectors in $\mathbb{R}^3$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero, $\mathbf{c} \cdot \mathbf{d} = 0$ and λ is a real parameter. Describe the geometrical relationship of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ to the relevant line, assuming that $\mathbf{d} \neq \mathbf{0}$.

Show that the solutions of (2) satisfy an equation of the form (1), defining $\mathbf{a}$, $\mathbf{b}$ and $\lambda(\mathbf{x})$ in terms of $\mathbf{c}$ and $\mathbf{d}$ such that $\mathbf{a} \cdot \mathbf{b} = 0$ and $|\mathbf{b}| = |\mathbf{c}|$. Deduce that the conditions on $\mathbf{c}$ and $\mathbf{d}$ are sufficient for (2) to have solutions.

For each of the lines described by (1) and (2), find the point $\mathbf{x}$ that is closest to a given fixed point $\mathbf{y}$.

Find the line of intersection of the two planes $\mathbf{x} \cdot \mathbf{m} = \mu$ and $\mathbf{x} \cdot \mathbf{n} = \nu$, where $\mathbf{m}$ and $\mathbf{n}$ are constant unit vectors, $\mathbf{m} \times \mathbf{n} \neq \mathbf{0}$ and μ and ν are constants. Express your answer in each of the forms (1) and (2), giving both $\mathbf{a}$ and $\mathbf{d}$ as linear combinations of $\mathbf{m}$ and $\mathbf{n}$.

6A Vectors and Matrices

The map $\Phi(\mathbf{x}) = \mathbf{n} \times (\mathbf{x} \times \mathbf{n}) + \alpha(\mathbf{n} \cdot \mathbf{x})\,\mathbf{n}$ is defined for $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$ and α is a constant.

(a) Find the inverse map Φ^{-1}, when it exists, and determine the values of α for which it does.

(b) When Φ is not invertible, find its image and kernel, and explain geometrically why these subspaces are perpendicular.

(c) Let $\mathbf{y} = \Phi(\mathbf{x})$. Find the components A_{ij} of the matrix A such that $y_i = A_{ij} x_j$. When Φ is invertible, find the components of the matrix B such that $x_i = B_{ij} y_j$.

(d) Now let A be as defined in (c) for the case $\mathbf{n} = \frac{1}{\sqrt{3}}(1,1,1)$, and let

$$C = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}.$$

By analysing a suitable determinant, for all values of α find all vectors $\mathbf{x}$ such that $A\mathbf{x} = C\mathbf{x}$. Explain your results by interpreting A and C geometrically.

7B Vectors and Matrices

(a) Find the eigenvalues and eigenvectors of the matrix

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -2 & 3 \end{pmatrix} .$$

(b) Under what conditions on the 3×3 matrix A and the vector $\mathbf{b}$ in $\mathbb{R}^3$ does the equation

$$A\mathbf{x} = \mathbf{b} \qquad (*)$$

have 0, 1, or infinitely many solutions for the vector $\mathbf{x}$ in $\mathbb{R}^3$? Give clear, concise arguments to support your answer, explaining why just these three possibilities are allowed.

(c) Using the results of (a), or otherwise, find all solutions to $(*)$ when

$$A = M - \lambda I \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

in each of the cases $\lambda = 0,\ 1,\ 2$.

8B Vectors and Matrices

(a) Let M be a real symmetric $n\times n$ matrix. Prove the following.

(i) Each eigenvalue of M is real.

(ii) Each eigenvector can be chosen to be real.

(iii) Eigenvectors with different eigenvalues are orthogonal.

(b) Let A be a real antisymmetric $n\times n$ matrix. Prove that each eigenvalue of A^2 is real and is less than or equal to zero.

If $-\lambda^2$ and $-\mu^2$ are distinct, non-zero eigenvalues of A^2, show that there exist orthonormal vectors $\mathbf{u}$, $\mathbf{u}'$, $\mathbf{w}$, $\mathbf{w}'$ with

$$A\mathbf{u} = \lambda\mathbf{u}' , \qquad A\mathbf{w} = \mu\mathbf{w}' ,$$
$$A\mathbf{u}' = -\lambda\mathbf{u} , \qquad A\mathbf{w}' = -\mu\mathbf{w} .$$

9F Analysis I

(a) State, without proof, the ratio test for the series $\sum_{n \geqslant 1} a_n$, where $a_n > 0$. Give examples, without proof, to show that, when $a_{n+1} < a_n$ and $a_{n+1}/a_n \to 1$, the series may converge or diverge.

(b) Prove that $\displaystyle\sum_{k=1}^{n-1} \frac{1}{k} \geqslant \log\, n$.

(c) Now suppose that $a_n > 0$ and that, for n large enough, $\dfrac{a_{n+1}}{a_n} \leqslant 1 - \dfrac{c}{n}$ where $c > 1$. Prove that the series $\sum_{n \geqslant 1} a_n$ converges.

[*You may find it helpful to prove the inequality* $\log\,(1-x) < -x$ *for* $0 < x < 1$.]

10E Analysis I

State and prove the Intermediate Value Theorem.

A *fixed point* of a function $f : X \to X$ is an $x \in X$ with $f(x) = x$. Prove that every continuous function $f : [0,1] \to [0,1]$ has a fixed point.

Answer the following questions with justification.

(i) Does every continuous function $f : (0,1) \to (0,1)$ have a fixed point?

(ii) Does every continuous function $f : \mathbb{R} \to \mathbb{R}$ have a fixed point?

(iii) Does every function $f : [0,1] \to [0,1]$ (not necessarily continuous) have a fixed point?

(iv) Let $f : [0,1] \to [0,1]$ be a continuous function with $f(0) = 1$ and $f(1) = 0$. Can f have exactly two fixed points?

11E Analysis I

For each of the following two functions $f : \mathbb{R} \to \mathbb{R}$, determine the set of points at which f is continuous, and also the set of points at which f is differentiable.

$$\text{(i) } f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \ , \end{cases}$$

$$\text{(ii) } f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \ . \end{cases}$$

By modifying the function in (i), or otherwise, find a function (not necessarily continuous) $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable at 0 and nowhere else.

Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that f is not differentiable at the points $1/2, 1/3, 1/4, \ldots$, but is differentiable at all other points.

12D Analysis I

State and prove the Fundamental Theorem of Calculus.

Let $f : [0,1] \to \mathbb{R}$ be integrable, and set $F(x) = \int_0^x f(t)\,\mathrm{d}t$ for $0 < x < 1$. Must F be differentiable?

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and set $g(x) = f'(x)$ for $0 \leqslant x \leqslant 1$. Must the Riemann integral of g from 0 to 1 exist?

END OF PAPER