UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 27 May 2010 9:00 am to 12:00 noon

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold Cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Vectors and Matrices

Let A be the matrix representing a linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the bases $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$ and $\{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ of $\mathbb{R}^m$, so that $\Phi(\mathbf{b}_i) = A_{ji}\mathbf{c}_j$. Let $\{\mathbf{b}'_1, \ldots, \mathbf{b}'_n\}$ be another basis of $\mathbb{R}^n$ and let $\{\mathbf{c}'_1, \ldots, \mathbf{c}'_m\}$ be another basis of $\mathbb{R}^m$. Show that the matrix A' representing Φ with respect to these new bases satisfies $A' = C^{-1} A\, B$ with matrices B and C which should be defined.

2C Vectors and Matrices

(a) The complex numbers z_1 and z_2 satisfy the equations

$$z_1^3 = 1\,, \qquad z_2^9 = 512\,.$$

What are the possible values of $|z_1 - z_2|$? Justify your answer.

(b) Show that $|z_1 + z_2| \leqslant |z_1| + |z_2|$ for all complex numbers z_1 and z_2 . Does the inequality $|z_1 + z_2| + |z_1 - z_2| \leqslant 2\max(|z_1|, |z_2|)$ hold for all complex numbers z_1 and z_2? Justify your answer with a proof or a counterexample.

3D Analysis I

Let $\sum_{n\geqslant 0} a_n z^n$ be a complex power series. State carefully what it means for the power series to have radius of convergence R, with $R \in [0, \infty]$.

Suppose the power series has radius of convergence R, with $0 < R < \infty$. Show that the sequence $|\,a_n z^n|$ is unbounded if $|z| > R$.

Find the radius of convergence of $\sum_{n\geqslant 1} z^n/n^3$.

4E Analysis I

Find the limit of each of the following sequences; justify your answers.

(i)

$$\frac{1+2+\ldots+n}{n^2}\ ;$$

(ii)

$$\sqrt[n]{n}\ ;$$

(iii)

$$(a^n+b^n)^{1/n} \qquad \text{with} \quad 0 < a \leqslant b\,.$$

SECTION II

5A Vectors and Matrices

Let A and B be real $n \times n$ matrices.

(i) Define the trace of A, $\operatorname{tr}(A)$, and show that $\operatorname{tr}\left(A^T B\right) = \operatorname{tr}\left(B^T A\right)$.

(ii) Show that $\operatorname{tr}\left(A^T A\right) \geqslant 0$, with $\operatorname{tr}\left(A^T A\right) = 0$ if and only if A is the zero matrix. Hence show that

$$\left(\operatorname{tr}(A^T B)\right)^2 \leqslant \operatorname{tr}\left(A^T A\right) \operatorname{tr}\left(B^T B\right) .$$

Under what condition on A and B is equality achieved?

(iii) Find a basis for the subspace of 2×2 matrices X such that

$$\operatorname{tr}\left(A^T X\right) = \operatorname{tr}\left(B^T X\right) = \operatorname{tr}\left(C^T X\right) = 0,$$

where $$A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

6C Vectors and Matrices

Let $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ be vectors in $\mathbb{R}^3$. Give a definition of the dot product, $\mathbf{a}_1 \cdot \mathbf{a}_2$, the cross product, $\mathbf{a}_1 \times \mathbf{a}_2$, and the triple product, $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3$. Explain what it means to say that the three vectors are *linearly independent*.

Let $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ be vectors in $\mathbb{R}^3$. Let S be a 3×3 matrix with entries $S_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j$. Show that

$$(\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3)(\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3) = \det(S).$$

Hence show that S is of maximal rank if and only if the sets of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are both linearly independent.

Now let $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ and $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$ be sets of vectors in $\mathbb{R}^n$, and let T be an $n \times n$ matrix with entries $T_{ij} = \mathbf{c}_i \cdot \mathbf{d}_j$. Is it the case that T is of maximal rank if and only if the sets of vectors $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ and $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_n\}$ are both linearly independent? Justify your answer with a proof or a counterexample.

Given an integer $n > 2$, is it always possible to find a set of vectors $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ in $\mathbb{R}^n$ with the property that every pair is linearly independent and that every triple is linearly dependent? Justify your answer.

7B Vectors and Matrices

Let A be a complex $n \times n$ matrix with an eigenvalue λ. Show directly from the definitions that:

(i) A^r has an eigenvalue λ^r for any integer $r \geqslant 1$; and

(ii) if A is invertible then $\lambda \neq 0$ and A^{-1} has an eigenvalue λ^{-1}.

For any complex $n \times n$ matrix A, let $\chi_A(t) = \det(A - tI)$. Using standard properties of determinants, show that:

(iii) $\chi_{A^2}(t^2) = \chi_A(t)\, \chi_A(-t)$; and

(iv) if A is invertible,

$$\chi_{A^{-1}}(t) = (\det A)^{-1}(-1)^n t^n \chi_A(t^{-1}) \, .$$

Explain, including justifications, the relationship between the eigenvalues of A and the polynomial $\chi_A(t)$.

If A^4 has an eigenvalue μ, does it follow that A has an eigenvalue λ with $\lambda^4 = \mu$? Give a proof or counterexample.

8B Vectors and Matrices

Let R be a real orthogonal 3×3 matrix with a real eigenvalue λ corresponding to some real eigenvector. Show algebraically that $\lambda = \pm 1$ and interpret this result geometrically.

Each of the matrices

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \qquad N = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} , \qquad P = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

has an eigenvalue $\lambda = 1$. Confirm this by finding as many independent eigenvectors as possible with this eigenvalue, for each matrix in turn.

Show that one of the matrices above represents a rotation, and find the axis and angle of rotation. Which of the other matrices represents a reflection, and why?

State, with brief explanations, whether the matrices M, N, P are diagonalisable (i) over the real numbers; (ii) over the complex numbers.

9E Analysis I

Determine whether the following series converge or diverge. Any tests that you use should be carefully stated.

(a)

$$\sum_{n\geqslant 1} \frac{n!}{n^n} \; ;$$

(b)

$$\sum_{n\geqslant 1} \frac{1}{n + (\log n)^2} \; ;$$

(c)

$$\sum_{n\geqslant 1} \frac{(-1)^n}{1+\sqrt{n}} \; ;$$

(d)

$$\sum_{n\geqslant 1} \frac{(-1)^n}{n\,(2+(-1)^n)} \; .$$

10F Analysis I

(a) State and prove Taylor's theorem with the remainder in Lagrange's form.

(b) Suppose that $e : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $e(0) = 1$ and $e'(x) = e(x)$ for all $x \in \mathbb{R}$. Use the result of (a) to prove that

$$e(x) = \sum_{n\geqslant 0} \frac{x^n}{n!} \quad \text{for all} \quad x \in \mathbb{R}.$$

[*No property of the exponential function may be assumed.*]

11D Analysis I

Define what it means for a bounded function $f : [a, \infty) \to \mathbb{R}$ to be Riemann integrable.

Show that a monotonic function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, where $-\infty < a < b < \infty$.

Prove that if $f : [1, \infty) \to \mathbb{R}$ is a decreasing function with $f(x) \to 0$ as $x \to \infty$, then $\sum_{n \geqslant 1} f(n)$ and $\int_1^\infty f(x)\, dx$ either both diverge or both converge.

Hence determine, for $\alpha \in \mathbb{R}$, when $\sum_{n \geqslant 1} n^\alpha$ converges.

12F Analysis I

(a) Let $n \geqslant 1$ and f be a function $\mathbb{R} \to \mathbb{R}$. Define carefully what it means for f to be n times differentiable at a point $x_0 \in \mathbb{R}$.

Set $\operatorname{sign}(x) = \begin{cases} x/|x|, & x \neq 0, \\ 0, & x = 0. \end{cases}$

Consider the function $f(x)$ on the real line, with $f(0) = 0$ and

$$f(x) = x^2 \operatorname{sign}(x) \left| \cos \frac{\pi}{x} \right|, \quad x \neq 0.$$

(b) Is $f(x)$ differentiable at $x = 0$?

(c) Show that $f(x)$ has points of non-differentiability in any neighbourhood of $x = 0$.

(d) Prove that, in any finite interval I, the derivative $f'(x)$, at the points $x \in I$ where it exists, is bounded: $|f'(x)| \leqslant C$ where C depends on I.

END OF PAPER