UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2010

List of Courses

Paper 1, Section I

3D Analysis I

Let $\sum_{n\geqslant 0} a_n z^n$ be a complex power series. State carefully what it means for the power series to have radius of convergence R, with $R \in [0, \infty]$.

Suppose the power series has radius of convergence R, with $0 < R < \infty$. Show that the sequence $|a_n z^n|$ is unbounded if $|z| > R$.

Find the radius of convergence of $\sum_{n\geqslant 1} z^n/n^3$.

Paper 1, Section I

4E Analysis I

Find the limit of each of the following sequences; justify your answers.

(i)
$$\frac{1+2+\ldots+n}{n^2}\;;$$

(ii)
$$\sqrt[n]{n}\;;$$

(iii)
$$(a^n+b^n)^{1/n} \qquad \text{with} \quad 0 < a \leqslant b\,.$$

Paper 1, Section II
9E Analysis I

Determine whether the following series converge or diverge. Any tests that you use should be carefully stated.

(a)
$$\sum_{n \geqslant 1} \frac{n!}{n^n} \;;$$

(b)
$$\sum_{n \geqslant 1} \frac{1}{n + (\log n)^2} \;;$$

(c)
$$\sum_{n \geqslant 1} \frac{(-1)^n}{1 + \sqrt{n}} \;;$$

(d)
$$\sum_{n \geqslant 1} \frac{(-1)^n}{n\,(2 + (-1)^n)} \;.$$

Paper 1, Section II
10F Analysis I

(a) State and prove Taylor's theorem with the remainder in Lagrange's form.

(b) Suppose that $e : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $e(0) = 1$ and $e'(x) = e(x)$ for all $x \in \mathbb{R}$. Use the result of (a) to prove that

$$e(x) = \sum_{n \geqslant 0} \frac{x^n}{n!} \quad \text{for all} \quad x \in \mathbb{R}.$$

[*No property of the exponential function may be assumed.*]

Paper 1, Section II
11D Analysis I

Define what it means for a bounded function $f : [a, \infty) \to \mathbb{R}$ to be Riemann integrable.

Show that a monotonic function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, where $-\infty < a < b < \infty$.

Prove that if $f : [1, \infty) \to \mathbb{R}$ is a decreasing function with $f(x) \to 0$ as $x \to \infty$, then $\sum_{n \geqslant 1} f(n)$ and $\int_1^\infty f(x)\, dx$ either both diverge or both converge.

Hence determine, for $\alpha \in \mathbb{R}$, when $\sum_{n \geqslant 1} n^\alpha$ converges.

Paper 1, Section II
12F Analysis I

(a) Let $n \geqslant 1$ and f be a function $\mathbb{R} \to \mathbb{R}$. Define carefully what it means for f to be n times differentiable at a point $x_0 \in \mathbb{R}$.

Set $\operatorname{sign}(x) = \begin{cases} x/|x|, & x \neq 0, \\ 0, & x = 0. \end{cases}$

Consider the function $f(x)$ on the real line, with $f(0) = 0$ and

$$f(x) = x^2 \operatorname{sign}(x) \left| \cos \frac{\pi}{x} \right|, \quad x \neq 0.$$

(b) Is $f(x)$ differentiable at $x = 0$?

(c) Show that $f(x)$ has points of non-differentiability in any neighbourhood of $x = 0$.

(d) Prove that, in any finite interval I, the derivative $f'(x)$, at the points $x \in I$ where it exists, is bounded: $|f'(x)| \leqslant C$ where C depends on I.

Paper 2, Section I

1A Differential Equations

Find the general solutions to the following difference equations for y_n, $n \in \mathbb{N}$.

$$\begin{aligned}
&\text{(i)} \quad & y_{n+3} - 3\,y_{n+1} + 2\,y_n &= 0, \\
&\text{(ii)} \quad & y_{n+3} - 3\,y_{n+1} + 2\,y_n &= 2^n, \\
&\text{(iii)} \quad & y_{n+3} - 3\,y_{n+1} + 2\,y_n &= (-2)^n, \\
&\text{(iv)} \quad & y_{n+3} - 3\,y_{n+1} + 2\,y_n &= (-2)^n + 2^n.
\end{aligned}$$

Paper 2, Section I

2A Differential Equations

Let $f(x,y) = g(u,v)$ where the variables $\{x,y\}$ and $\{u,v\}$ are related by a smooth, invertible transformation. State the chain rule expressing the derivatives $\dfrac{\partial g}{\partial u}$ and $\dfrac{\partial g}{\partial v}$ in terms of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ and use this to deduce that

$$\frac{\partial^2 g}{\partial u\,\partial v} = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\frac{\partial^2 f}{\partial y^2} + H\frac{\partial f}{\partial x} + K\frac{\partial f}{\partial y}$$

where H and K are second-order partial derivatives, to be determined.

Using the transformation $x = uv$ and $y = u/v$ in the above identity, or otherwise, find the general solution of

$$x\,\frac{\partial^2 f}{\partial x^2} - \frac{y^2}{x}\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x} - \frac{y}{x}\frac{\partial f}{\partial y} = 0\,.$$

Paper 2, Section II

5A Differential Equations

(a) Consider the differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 , \qquad (1)$$

with $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{R}$. Show that $y(x) = e^{\lambda x}$ is a solution if and only if $p(\lambda) = 0$ where

$$p(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_2 \lambda^2 + a_1 \lambda + a_0 .$$

Show further that $y(x) = x e^{\mu x}$ is also a solution of (1) if μ is a root of the polynomial $p(\lambda)$ of multiplicity at least 2.

(b) By considering $v(t) = \dfrac{d^2 u}{dt^2}$, or otherwise, find the general real solution for $u(t)$ satisfying

$$\frac{d^4 u}{dt^4} + 2 \frac{d^2 u}{dt^2} = 4t^2 . \qquad (2)$$

By using a substitution of the form $u(t) = y(t^2)$ in (2), or otherwise, find the general real solution for $y(x)$, with x positive, where

$$4x^2 \frac{d^4 y}{dx^4} + 12x \frac{d^3 y}{dx^3} + (3 + 2x) \frac{d^2 y}{dx^2} + \frac{dy}{dx} = x .$$

Paper 2, Section II

6A Differential Equations

(a) By using a power series of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

or otherwise, find the general solution of the differential equation

$$xy'' - (1-x)y' - y = 0. \tag{1}$$

(b) Define the Wronskian $W(x)$ for a second order linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

and show that $W' + p(x)W = 0$. Given a non-trivial solution $y_1(x)$ of (2) show that $W(x)$ can be used to find a second solution $y_2(x)$ of (2) and give an expression for $y_2(x)$ in the form of an integral.

(c) Consider the equation (2) with

$$p(x) = -\frac{P(x)}{x} \qquad \text{and} \qquad q(x) = -\frac{Q(x)}{x}$$

where P and Q have Taylor expansions

$$P(x) = P_0 + P_1 x + \ldots, \qquad Q(x) = Q_0 + Q_1 x + \ldots$$

with P_0 a positive integer. Find the roots of the indicial equation for (2) with these assumptions. If $y_1(x) = 1 + \beta x + \ldots$ is a solution, use the method of part (b) to find the first two terms in a power series expansion of a linearly independent solution $y_2(x)$, expressing the coefficients in terms of P_0, P_1 and β.

Paper 2, Section II
7A Differential Equations

(a) Find the general solution of the system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \qquad (1)$$

(b) Depending on the parameter $\lambda \in \mathbb{R}$, find the general solution of the system of differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2 \begin{pmatrix} -\lambda \\ 1 \\ \lambda \end{pmatrix} e^{2t}, \qquad (2)$$

and explain why (2) has a particular solution of the form $\mathbf{c}e^{2t}$ with constant vector $\mathbf{c} \in \mathbb{R}^3$ for $\lambda = 1$ but not for $\lambda \neq 1$.

[*Hint: decompose* $\begin{pmatrix} -\lambda \\ 1 \\ \lambda \end{pmatrix}$ *in terms of the eigenbasis of the matrix in* (1).]

(c) For $\lambda = -1$, find the solution of (2) which goes through the point $(0, 1, 0)$ at $t = 0$.

Paper 2, Section II
8A Differential Equations

(a) State how the nature of a critical (or stationary) point of a function $f(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$ can be determined by consideration of the eigenvalues of the Hessian matrix H of $f(\mathbf{x})$, assuming H is non-singular.

(b) Let $f(x, y) = xy(1 - x - y)$. Find all the critical points of the function $f(x, y)$ and determine their nature. Determine the zero contour of $f(x, y)$ and sketch a contour plot showing the behaviour of the contours in the neighbourhood of the critical points.

(c) Now let $g(x, y) = x^3y^2(1 - x - y)$. Show that $(0, 1)$ is a critical point of $g(x, y)$ for which the Hessian matrix of g is singular. Find an approximation for $g(x, y)$ to lowest non-trivial order in the neighbourhood of the point $(0, 1)$. Does g have a maximum or a minimum at $(0, 1)$? Justify your answer.

Paper 4, Section I
3B Dynamics and Relativity

A particle of mass m and charge q moves with trajectory $\mathbf{r}(t)$ in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. Write down the Lorentz force on the particle and use Newton's Second Law to deduce that

$$\dot{\mathbf{r}} - \omega\,\mathbf{r} \times \hat{\mathbf{z}} = \mathbf{c}\,,$$

where $\mathbf{c}$ is a constant vector and ω is to be determined. Find $\mathbf{c}$ and hence $\mathbf{r}(t)$ for the initial conditions

$$\mathbf{r}(0) = a\hat{\mathbf{x}} \qquad \text{and} \qquad \dot{\mathbf{r}}(0) = u\hat{\mathbf{y}} + v\hat{\mathbf{z}}$$

where a, u and v are constants. Sketch the particle's trajectory in the case $a\omega + u = 0$.

[*Unit vectors* $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ *correspond to a set of Cartesian coordinates.*]

Paper 4, Section I
4B Dynamics and Relativity

Let S be an inertial frame with coordinates (t, x) in two-dimensional spacetime. Write down the Lorentz transformation giving the coordinates (t', x') in a second inertial frame S' moving with velocity v relative to S. If a particle has constant velocity u in S, find its velocity u' in S'. Given that $|u| < c$ and $|v| < c$, show that $|u'| < c$.

Paper 4, Section II
9B Dynamics and Relativity

A sphere of uniform density has mass m and radius a. Find its moment of inertia about an axis through its centre.

A marble of uniform density is released from rest on a plane inclined at an angle α to the horizontal. Let the time taken for the marble to travel a distance ℓ down the plane be: (i) t_1 if the plane is perfectly smooth; or (ii) t_2 if the plane is rough and the marble rolls without slipping.

Explain, with a clear discussion of the forces acting on the marble, whether or not its energy is conserved in each of the cases (i) and (ii). Show that $t_1/t_2 = \sqrt{5/7}$.

Suppose that the original marble is replaced by a new one with the same mass and radius but with a hollow centre, so that its moment of inertia is $\lambda m a^2$ for some constant λ. What is the new value for t_1/t_2?

Paper 4, Section II

10B Dynamics and Relativity

A particle of unit mass moves in a plane with polar coordinates (r, θ) and components of acceleration $(\ddot{r} - r\dot{\theta}^2,\ r\ddot{\theta} + 2\dot{r}\dot{\theta})$. The particle experiences a force corresponding to a potential $-Q/r$. Show that

$$E = \frac{1}{2}\dot{r}^2 + U(r) \qquad \text{and} \qquad h = r^2\dot{\theta}$$

are constants of the motion, where

$$U(r) = \frac{h^2}{2r^2} - \frac{Q}{r}\,.$$

Sketch the graph of $U(r)$ in the cases $Q > 0$ and $Q < 0$.

(a) Assuming $Q > 0$ and $h > 0$, for what range of values of E do bounded orbits exist? Find the minimum and maximum distances from the origin, $r_{\rm min}$ and $r_{\rm max}$, on such an orbit and show that

$$r_{\rm min} + r_{\rm max} = \frac{Q}{|E|}\,.$$

Prove that the minimum and maximum values of the particle's speed, $v_{\rm min}$ and $v_{\rm max}$, obey

$$v_{\rm min} + v_{\rm max} = \frac{2Q}{h}\,.$$

(b) Now consider trajectories with $E > 0$ and Q of either sign. Find the distance of closest approach, $r_{\rm min}$, in terms of the impact parameter, b, and v_∞, the limiting value of the speed as $r \to \infty$. Deduce that if $b \ll |Q|/v_\infty^2$ then, to leading order,

$$r_{\rm min} \approx \frac{2|Q|}{v_\infty^2} \quad \text{for} \quad Q < 0\,, \qquad\qquad r_{\rm min} \approx \frac{b^2 v_\infty^2}{2Q} \quad \text{for} \quad Q > 0\,.$$

Paper 4, Section II
11B Dynamics and Relativity

Consider a set of particles with position vectors $\mathbf{r}_i(t)$ and masses m_i, where $i = 1, 2, \ldots, N$. Particle i experiences an external force $\mathbf{F}_i$ and an internal force $\mathbf{F}_{ij}$ from particle j, for each $j \neq i$. Stating clearly any assumptions you need, show that

$$\frac{d\mathbf{P}}{dt} = \mathbf{F} \qquad \text{and} \qquad \frac{d\mathbf{L}}{dt} = \mathbf{G},$$

where $\mathbf{P}$ is the total momentum, $\mathbf{F}$ is the total external force, $\mathbf{L}$ is the total angular momentum about a fixed point $\mathbf{a}$, and $\mathbf{G}$ is the total external torque about $\mathbf{a}$.

Does the result $\dfrac{d\mathbf{L}}{dt} = \mathbf{G}$ still hold if the fixed point $\mathbf{a}$ is replaced by the centre of mass of the system? Justify your answer.

Suppose now that the external force on particle i is $-k\dfrac{d\mathbf{r}_i}{dt}$ and that all the particles have the same mass m. Show that

$$\mathbf{L}(t) = \mathbf{L}(0)\, e^{-kt/m} .$$

Paper 4, Section II
12B Dynamics and Relativity

A particle A of rest mass m is fired at an identical particle B which is stationary in the laboratory. On impact, A and B annihilate and produce two massless photons whose energies are equal. Assuming conservation of four-momentum, show that the angle θ between the photon trajectories is given by

$$\cos\theta = \frac{E - 3mc^2}{E + mc^2}$$

where E is the relativistic energy of A.

Let v be the speed of the incident particle A. For what value of v/c will the photons move in perpendicular directions? If v is very small compared with c, show that

$$\theta \approx \pi - v/c .$$

[*All quantities referred to are measured in the laboratory frame.*]

Paper 3, Section I
1D Groups

Write down the matrix representing the following transformations of $\mathbb{R}^3$:

(i) clockwise rotation of $45°$ around the x axis,

(ii) reflection in the plane $x = y$,

(iii) the result of first doing (i) and then (ii).

Paper 3, Section I
2D Groups

Express the element $(123)(234)$ in S_5 as a product of disjoint cycles. Show that it is in A_5. Write down the elements of its conjugacy class in A_5.

Paper 3, Section II
5D Groups

(i) State the orbit-stabilizer theorem.

Let G be the group of rotations of the cube, X the set of faces. Identify the stabilizer of a face, and hence compute the order of G.

Describe the orbits of G on the set $X \times X$ of pairs of faces.

(ii) Define what it means for a subgroup N of G to be *normal*. Show that G has a normal subgroup of order 4.

Paper 3, Section II
6D Groups

State Lagrange's theorem. Let p be a prime number. Prove that every group of order p is cyclic. Prove that every abelian group of order p^2 is isomorphic to either $C_p \times C_p$ or C_{p^2}.

Show that D_{12}, the dihedral group of order 12, is not isomorphic to the alternating group A_4.

Paper 3, Section II
7D Groups

Let G be a group, X a set on which G acts transitively, B the stabilizer of a point $x \in X$.

Show that if $g \in G$ stabilizes the point $y \in X$, then there exists an $h \in G$ with $hgh^{-1} \in B$.

Let $G = SL_2(\mathbb{C})$, acting on $\mathbb{C} \cup \{\infty\}$ by Möbius transformations. Compute $B = G_\infty$, the stabilizer of ∞. Given

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

compute the set of fixed points $\Big\{x \in \mathbb{C} \cup \{\infty\} \,\Big|\, gx = x\Big\}$.

Show that every element of G is conjugate to an element of B.

Paper 3, Section II
8D Groups

Let G be a finite group, X the set of proper subgroups of G. Show that conjugation defines an action of G on X.

Let B be a proper subgroup of G. Show that the orbit of G on X containing B has size at most the index $|G : B|$. Show that there exists a $g \in G$ which is not conjugate to an element of B.

Paper 4, Section I
1E Numbers and Sets

(a) Find the smallest residue x which equals $28!\,13^{28} \pmod{31}$.

[*You may use any standard theorems provided you state them correctly.*]

(b) Find all integers x which satisfy the system of congruences

$$\begin{aligned} x &\equiv 1 \pmod{2}, \\ 2x &\equiv 1 \pmod{3}, \\ 2x &\equiv 4 \pmod{10}, \\ x &\equiv 10 \pmod{67}. \end{aligned}$$

Paper 4, Section I
2E Numbers and Sets

(a) Let r be a real root of the polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, with integer coefficients a_i and leading coefficient 1. Show that if r is rational, then r is an integer.

(b) Write down a series for e. By considering $q!e$ for every natural number q, show that e is irrational.

Paper 4, Section II
5E Numbers and Sets

The Fibonacci numbers F_n are defined for all natural numbers n by the rules

$$F_1 = 1, \qquad F_2 = 1, \qquad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geqslant 3.$$

Prove by induction on k that, for any n,

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n \quad \text{for all } k \geqslant 2.$$

Deduce that

$$F_{2n} = F_n(F_{n+1} + F_{n-1}) \qquad \text{for all } n \geqslant 2.$$

Put $L_1 = 1$ and $L_n = F_{n+1} + F_{n-1}$ for $n > 1$. Show that these (Lucas) numbers L_n satisfy

$$L_1 = 1, \qquad L_2 = 3, \qquad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geqslant 3.$$

Show also that, for all n, the greatest common divisor (F_n, F_{n+1}) is 1, and that the greatest common divisor (F_n, L_n) is at most 2.

Paper 4, Section II
6E Numbers and Sets

State and prove Fermat's Little Theorem.

Let p be an odd prime. If $p \neq 5$, show that p divides $10^n - 1$ for infinitely many natural numbers n.

Hence show that p divides infinitely many of the integers

$$5, \qquad 55, \qquad 555, \qquad 5555, \qquad \ldots \; .$$

Paper 4, Section II
7E Numbers and Sets

(a) Let A, B be finite non–empty sets, with $|A| = a$, $|B| = b$. Show that there are b^a mappings from A to B. How many of these are injective ?

(b) State the Inclusion–Exclusion principle.

(c) Prove that the number of surjective mappings from a set of size n onto a set of size k is

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n \qquad \text{for } n \geqslant k \geqslant 1\,.$$

Deduce that

$$n! = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n\,.$$

Paper 4, Section II
8E Numbers and Sets

What does it mean for a set to be countable ?

Show that $\mathbb{Q}$ is countable, but $\mathbb{R}$ is not. Show also that the union of two countable sets is countable.

A subset A of $\mathbb{R}$ has the property that, given $\epsilon > 0$ and $x \in \mathbb{R}$, there exist reals a, b with $a \in A$ and $b \notin A$ with $|x-a| < \epsilon$ and $|x-b| < \epsilon$. Can A be countable ? Can A be uncountable ? Justify your answers.

A subset B of $\mathbb{R}$ has the property that given $b \in B$ there exists $\epsilon > 0$ such that if $0 < |b-x| < \epsilon$ for some $x \in \mathbb{R}$, then $x \notin B$. Is B countable ? Justify your answer.

Paper 2, Section I
3F Probability

Jensen's inequality states that for a convex function f and a random variable X with a finite mean, $\mathbb{E}f(X) \geqslant f\big(\mathbb{E}X\big)$.

(a) Suppose that $f(x) = x^m$ where m is a positive integer, and X is a random variable taking values $x_1, \ldots, x_N \geqslant 0$ with equal probabilities, and where the sum $x_1 + \ldots + x_N = 1$. Deduce from Jensen's inequality that

$$\sum_{i=1}^{N} f(x_i) \geqslant N f\left(\frac{1}{N}\right). \tag{1}$$

(b) N horses take part in m races. The results of different races are independent. The probability for horse i to win any given race is $p_i \geqslant 0$, with $p_1 + \ldots + p_N = 1$.

Let Q be the probability that a single horse wins all m races. Express Q as a polynomial of degree m in the variables $p_1, \ldots, p_N$.

By using (1) or otherwise, prove that $Q \geqslant N^{1-m}$.

Paper 2, Section I
4F Probability

Let X and Y be two non-constant random variables with finite variances. The correlation coefficient $\rho(X,Y)$ is defined by

$$\rho(X,Y) = \frac{\mathbb{E}\big[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\big]}{\big(\mathrm{Var}\ X\big)^{1/2}\big(\mathrm{Var}\ Y\big)^{1/2}}.$$

(a) Using the Cauchy–Schwarz inequality or otherwise, prove that

$$-1 \leqslant \rho(X,Y) \leqslant 1\,.$$

(b) What can be said about the relationship between X and Y when either (i) $\rho(X,Y) = 0$ or (ii) $|\rho(X,Y)| = 1$. [*Proofs are not required.*]

(c) Take $0 \leqslant r \leqslant 1$ and let X, X' be independent random variables taking values ± 1 with probabilities $1/2$. Set

$$Y = \begin{cases} X, & \text{with probability } r, \\ X', & \text{with probability } 1-r. \end{cases}$$

Find $\rho(X,Y)$.

Paper 2, Section II
9F Probability

(a) What does it mean to say that a random variable X with values $n = 1, 2, \ldots$ has a geometric distribution with a parameter p where $p \in (0, 1)$?

An expedition is sent to the Himalayas with the objective of catching a pair of wild yaks for breeding. Assume yaks are loners and roam about the Himalayas at random. The probability $p \in (0, 1)$ that a given trapped yak is male is independent of prior outcomes. Let N be the number of yaks that must be caught until a breeding pair is obtained.

(b) Find the expected value of N.

(c) Find the variance of N.

Paper 2, Section II
10F Probability

The yearly levels of water in the river Camse are independent random variables X_1, X_2, ..., with a given continuous distribution function $F(x) = \mathbb{P}(X_i \leqslant x)$, $x \geqslant 0$ and $F(0) = 0$. The levels have been observed in years 1, ..., n and their values X_1, ..., X_n recorded. The local council has decided to construct a dam of height

$$Y_n = \max\ \big[X_1, \ldots, X_n\big].$$

Let τ be the subsequent time that elapses before the dam overflows:

$$\tau = \min\ \big[t \geqslant 1 :\ X_{n+t} > Y_n\big].$$

(a) Find the distribution function $\mathbb{P}(Y_n \leqslant z)$, $z > 0$, and show that the mean value $\mathbb{E}Y_n = \displaystyle\int_0^\infty [1 - F(z)^n]\mathrm{d}z$.

(b) Express the conditional probability $\mathbb{P}(\tau = k \,|\, Y_n = z)$, where $k = 1, 2, \ldots$ and $z > 0$, in terms of F.

(c) Show that the unconditional probability

$$\mathbb{P}(\tau = k) = \frac{n}{(k + n - 1)(k + n)}, \quad k = 1, 2, \ldots.$$

(d) Determine the mean value $\mathbb{E}\,\tau$.

Paper 2, Section II

11F Probability

In a branching process every individual has probability p_k of producing exactly k offspring, $k = 0, 1, \ldots$, and the individuals of each generation produce offspring independently of each other and of individuals in preceding generations. Let X_n represent the size of the nth generation. Assume that $X_0 = 1$ and $p_0 > 0$ and let $F_n(s)$ be the generating function of X_n. Thus

$$F_1(s) = \mathbb{E}s^{X_1} = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \leqslant 1.$$

(a) Prove that

$$F_{n+1}(s) = F_n(F_1(s)).$$

(b) State a result in terms of $F_1(s)$ about the probability of eventual extinction. [*No proofs are required.*]

(c) Suppose the probability that an individual leaves k descendants in the next generation is $p_k = 1/2^{k+1}$, for $k \geqslant 0$. Show from the result you state in (b) that extinction is certain. Prove further that in this case

$$F_n(s) = \frac{n - (n-1)s}{(n+1) - ns}, \quad n \geqslant 1,$$

and deduce the probability that the nth generation is empty.

Paper 2, Section II

12F Probability

Let X_1, X_2 be bivariate normal random variables, with the joint probability density function

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{\varphi(x_1,x_2)}{2(1-\rho^2)}\right],$$

where

$$\varphi(x_1,x_2) = \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2$$

and $x_1, x_2 \in \mathbb{R}$.

(a) Deduce that the marginal probability density function

$$f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right].$$

(b) Write down the moment-generating function of X_2 in terms of μ_2 and σ_2. [*No proofs are required.*]

(c) By considering the ratio $f_{X_1,X_2}(x_1,x_2)/f_{X_2}(x_2)$ prove that, conditional on $X_2 = x_2$, the distribution of X_1 is normal, with mean and variance $\mu_1 + \rho\sigma_1(x_2-\mu_2)/\sigma_2$ and $\sigma_1^2(1-\rho^2)$, respectively.

Paper 3, Section I

3C Vector Calculus

Consider the vector field

$$\mathbf{F} = (-y/(x^2+y^2),\, x/(x^2+y^2),\, 0)$$

defined on all of $\mathbb{R}^3$ except the z axis. Compute $\boldsymbol{\nabla}\times\mathbf{F}$ on the region where it is defined.

Let γ_1 be the closed curve defined by the circle in the xy-plane with centre $(2,2,0)$ and radius 1, and γ_2 be the closed curve defined by the circle in the xy-plane with centre $(0,0,0)$ and radius 1.

By using your earlier result, or otherwise, evaluate the line integral $\oint_{\gamma_1}\mathbf{F}\cdot d\mathbf{x}$.

By explicit computation, evaluate the line integral $\oint_{\gamma_2}\mathbf{F}\cdot d\mathbf{x}$. Is your result consistent with Stokes' theorem? Explain your answer briefly.

Paper 3, Section I

4C Vector Calculus

A curve in two dimensions is defined by the parameterised Cartesian coordinates

$$x(u) = ae^{bu}\cos u\,, \qquad y(u) = ae^{bu}\sin u\,,$$

where the constants $a, b > 0$. Sketch the curve segment corresponding to the range $0 \leqslant u \leqslant 3\pi$. What is the length of the curve segment between the points $(x(0), y(0))$ and $(x(U), y(U))$, as a function of U?

A geometrically sensitive ant walks along the curve with varying speed $\kappa(u)^{-1}$, where $\kappa(u)$ is the curvature at the point corresponding to parameter u. Find the time taken by the ant to walk from $(x(2n\pi), y(2n\pi))$ to $(x(2(n+1)\pi), y(2(n+1)\pi))$, where n is a positive integer, and hence verify that this time is independent of n.

[*You may quote without proof the formula* $\kappa(u) = \dfrac{|x'(u)y''(u) - y'(u)x''(u)|}{((x'(u))^2 + (y'(u))^2)^{3/2}}$.]

Paper 3, Section II

9C Vector Calculus

(a) Define a rank two tensor and show that if two rank two tensors A_{ij} and B_{ij} are the same in one Cartesian coordinate system, then they are the same in all Cartesian coordinate systems.

The quantity C_{ij} has the property that, for every rank two tensor A_{ij}, the quantity $C_{ij}A_{ij}$ is a scalar. Is C_{ij} necessarily a rank two tensor? Justify your answer with a proof from first principles, or give a counterexample.

(b) Show that, if a tensor T_{ij} is invariant under rotations about the x_3-axis, then it has the form

$$\begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} .$$

(c) The *inertia tensor* about the origin of a rigid body occupying volume V and with variable mass density $\rho(\mathbf{x})$ is defined to be

$$I_{ij} = \int_V \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j)\,\mathrm{d}V .$$

The rigid body B has uniform density ρ and occupies the cylinder

$$\{\,(x_1, x_2, x_3) \,:\, -2 \leqslant x_3 \leqslant 2,\; x_1^2 + x_2^2 \leqslant 1\,\} .$$

Show that the inertia tensor of B about the origin is diagonal in the (x_1, x_2, x_3) coordinate system, and calculate its diagonal elements.

Paper 3, Section II
10C Vector Calculus

Let $f(x,y)$ be a function of two variables, and R a region in the xy-plane. State the rule for evaluating $\int_R f(x,y)\,\mathrm{d}x\mathrm{d}y$ as an integral with respect to new variables $u(x,y)$ and $v(x,y)$.

Sketch the region R in the xy-plane defined by

$$R = \{ (x,y) : x^2+y^2 \leqslant 2\,,\ x^2-y^2 \geqslant 1\,,\ x \geqslant 0\,,\ y \geqslant 0 \}\,.$$

Sketch the corresponding region in the uv-plane, where

$$u = x^2+y^2\,, \qquad v = x^2-y^2\,.$$

Express the integral

$$I = \int_R (x^5y - xy^5)\exp(4x^2y^2)\,\mathrm{d}x\,\mathrm{d}y$$

as an integral with respect to u and v. Hence, or otherwise, calculate I.

Paper 3, Section II
11C Vector Calculus

State the divergence theorem (also known as Gauss' theorem) relating the surface and volume integrals of appropriate fields.

The surface S_1 is defined by the equation $z = 3-2x^2-2y^2$ for $1 \leqslant z \leqslant 3$; the surface S_2 is defined by the equation $x^2+y^2=1$ for $0 \leqslant z \leqslant 1$; the surface S_3 is defined by the equation $z=0$ for x,y satisfying $x^2+y^2 \leqslant 1$. The surface S is defined to be the union of the surfaces S_1, S_2 and S_3. Sketch the surfaces S_1, S_2, S_3 and (hence) S.

The vector field $\mathbf{F}$ is defined by

$$\mathbf{F}(x,y,z) = (\,xy+x^6,\ -\tfrac{1}{2}y^2+y^8,\ z\,)\,.$$

Evaluate the integral

$$\oint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}\,,$$

where the surface element $\mathrm{d}\mathbf{S}$ points in the direction of the outward normal to S.

Paper 3, Section II

12C Vector Calculus

Given a spherically symmetric mass distribution with density ρ, explain how to obtain the gravitational field $\mathbf{g} = -\boldsymbol{\nabla}\phi$, where the potential ϕ satisfies Poisson's equation

$$\nabla^2\phi = 4\pi G\rho\,.$$

The remarkable planet Geometria has radius 1 and is composed of an infinite number of stratified spherical shells S_n labelled by integers $n \geqslant 1$. The shell S_n has uniform density $2^{n-1}\rho_0$, where ρ_0 is a constant, and occupies the volume between radius 2^{-n+1} and 2^{-n}.

Obtain a closed form expression for the mass of Geometria.

Obtain a closed form expression for the gravitational field $\mathbf{g}$ due to Geometria at a distance $r = 2^{-N}$ from its centre of mass, for each positive integer $N \geqslant 1$. What is the potential $\phi(r)$ due to Geometria for $r > 1$?

Paper 1, Section I
1A Vectors and Matrices

Let A be the matrix representing a linear map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the bases $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$ and $\{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ of $\mathbb{R}^m$, so that $\Phi(\mathbf{b}_i) = A_{ji}\mathbf{c}_j$. Let $\{\mathbf{b}'_1, \ldots, \mathbf{b}'_n\}$ be another basis of $\mathbb{R}^n$ and let $\{\mathbf{c}'_1, \ldots, \mathbf{c}'_m\}$ be another basis of $\mathbb{R}^m$. Show that the matrix A' representing Φ with respect to these new bases satisfies $A' = C^{-1}A\,B$ with matrices B and C which should be defined.

Paper 1, Section I
2C Vectors and Matrices

(a) The complex numbers z_1 and z_2 satisfy the equations

$$z_1^3 = 1\,, \qquad z_2^9 = 512\,.$$

What are the possible values of $|z_1 - z_2|$? Justify your answer.

(b) Show that $|z_1 + z_2| \leqslant |z_1| + |z_2|$ for all complex numbers z_1 and z_2 . Does the inequality $|z_1 + z_2| + |z_1 - z_2| \leqslant 2\max(|z_1|, |z_2|)$ hold for all complex numbers z_1 and z_2? Justify your answer with a proof or a counterexample.

Paper 1, Section II
5A Vectors and Matrices

Let A and B be real $n \times n$ matrices.

(i) Define the trace of A, $\operatorname{tr}(A)$, and show that $\operatorname{tr}\left(A^T B\right) = \operatorname{tr}\left(B^T A\right)$.

(ii) Show that $\operatorname{tr}\left(A^T A\right) \geqslant 0$, with $\operatorname{tr}\left(A^T A\right) = 0$ if and only if A is the zero matrix. Hence show that

$$\left(\operatorname{tr}(A^T B)\right)^2 \leqslant \operatorname{tr}\left(A^T A\right) \operatorname{tr}\left(B^T B\right)\ .$$

Under what condition on A and B is equality achieved?

(iii) Find a basis for the subspace of 2×2 matrices X such that

$$\operatorname{tr}\left(A^T X\right) = \operatorname{tr}\left(B^T X\right) = \operatorname{tr}\left(C^T X\right) = 0\,,$$

where $\quad A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$

Paper 1, Section II

6C Vectors and Matrices

Let $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ be vectors in $\mathbb{R}^3$. Give a definition of the dot product, $\mathbf{a}_1 \cdot \mathbf{a}_2$, the cross product, $\mathbf{a}_1 \times \mathbf{a}_2$, and the triple product, $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3$. Explain what it means to say that the three vectors are *linearly independent.*

Let $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ be vectors in $\mathbb{R}^3$. Let S be a 3×3 matrix with entries $S_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j$. Show that

$$(\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3)(\mathbf{b}_1 \cdot \mathbf{b}_2 \times \mathbf{b}_3) \;=\; \det(S)\,.$$

Hence show that S is of maximal rank if and only if the sets of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are both linearly independent.

Now let $\{\mathbf{c}_1,\, \mathbf{c}_2,\, \ldots,\, \mathbf{c}_n\}$ and $\{\mathbf{d}_1,\, \mathbf{d}_2,\, \ldots,\, \mathbf{d}_n\}$ be sets of vectors in $\mathbb{R}^n$, and let T be an $n \times n$ matrix with entries $T_{ij} = \mathbf{c}_i \cdot \mathbf{d}_j$. Is it the case that T is of maximal rank if and only if the sets of vectors $\{\mathbf{c}_1,\, \mathbf{c}_2,\, \ldots,\, \mathbf{c}_n\}$ and $\{\mathbf{d}_1,\, \mathbf{d}_2,\, \ldots,\, \mathbf{d}_n\}$ are both linearly independent? Justify your answer with a proof or a counterexample.

Given an integer $n > 2$, is it always possible to find a set of vectors $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ in $\mathbb{R}^n$ with the property that every pair is linearly independent and that every triple is linearly dependent? Justify your answer.

Paper 1, Section II

7B Vectors and Matrices

Let A be a complex $n \times n$ matrix with an eigenvalue λ. Show directly from the definitions that:

(i) A^r has an eigenvalue λ^r for any integer $r \geqslant 1$; and

(ii) if A is invertible then $\lambda \neq 0$ and A^{-1} has an eigenvalue λ^{-1}.

For any complex $n \times n$ matrix A, let $\chi_A(t) = \det(A - tI)$. Using standard properties of determinants, show that:

(iii) $\chi_{A^2}(t^2) = \chi_A(t)\,\chi_A(-t)$; and

(iv) if A is invertible,

$$\chi_{A^{-1}}(t) = (\det A)^{-1}(-1)^n t^n \chi_A(t^{-1})\,.$$

Explain, including justifications, the relationship between the eigenvalues of A and the polynomial $\chi_A(t)$.

If A^4 has an eigenvalue μ, does it follow that A has an eigenvalue λ with $\lambda^4 = \mu$? Give a proof or counterexample.

Paper 1, Section II

8B Vectors and Matrices

Let R be a real orthogonal $3{\times}3$ matrix with a real eigenvalue λ corresponding to some real eigenvector. Show algebraically that $\lambda = \pm 1$ and interpret this result geometrically.

Each of the matrices

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad P = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

has an eigenvalue $\lambda = 1$. Confirm this by finding as many independent eigenvectors as possible with this eigenvalue, for each matrix in turn.

Show that one of the matrices above represents a rotation, and find the axis and angle of rotation. Which of the other matrices represents a reflection, and why?

State, with brief explanations, whether the matrices M, N, P are diagonalisable (i) over the real numbers; (ii) over the complex numbers.