UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday, 2 June, 2009 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***B*** *and* ***D*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

Show that every orthogonal 2×2 matrix R is the product of at most two reflections in lines through the origin.

Every isometry of the Euclidean plane $\mathbb{R}^2$ can be written as the composition of an orthogonal matrix and a translation. Deduce from this that every isometry of the Euclidean plane $\mathbb{R}^2$ is a product of reflections.

Give an example of an isometry of $\mathbb{R}^2$ that is not the product of fewer than three reflections. Justify your answer.

2D Groups

State and prove Lagrange's theorem. Give an example to show that an integer k may divide the order of a group G without there being a subgroup of order k.

3B Vector Calculus

What does it mean for a vector field $\mathbf{F}$ to be *irrotational* ?

The field $\mathbf{F}$ is irrotational and $\mathbf{x}_0$ is a given point. Write down a scalar potential $V(\mathbf{x})$ with $\mathbf{F} = -\boldsymbol{\nabla} V$ and $V(\mathbf{x}_0) = 0$. Show that this potential is well defined.

For what value of m is the field $\dfrac{\cos\theta\cos\phi}{r}\mathbf{e}_\theta + \dfrac{m\sin\phi}{r}\mathbf{e}_\phi$ irrotational, where (r, θ, ϕ) are spherical polar coordinates? What is the corresponding potential $V(\mathbf{x})$ when $\mathbf{x}_0$ is the point $r = 1, \theta = 0$?

$$\left[\textit{In spherical polar coordinates } \boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}.\right]$$

4B **Vector Calculus**

State the value of $\partial x_i/\partial x_j$ and find $\partial r/\partial x_j$, where $r = |\mathbf{x}|$.

A vector field $\mathbf{u}$ is given by

$$\mathbf{u} = \frac{\mathbf{k}}{r} + \frac{(\mathbf{k}\cdot\mathbf{x})\mathbf{x}}{r^3},$$

where $\mathbf{k}$ is a constant vector. Calculate the second-rank tensor $d_{ij} = \partial u_i/\partial x_j$ using suffix notation, and show that d_{ij} splits naturally into symmetric and antisymmetric parts. Deduce that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and that

$$\boldsymbol{\nabla}\times\mathbf{u} = \frac{2\mathbf{k}\times\mathbf{x}}{r^3}\ .$$

SECTION II

5D Groups

State and prove the orbit–stabilizer theorem.

Let G be the group of all symmetries of a regular octahedron, including both orientation-preserving and orientation-reversing symmetries. How many symmetries are there in the group G? Let D be the set of straight lines that join a vertex of the octahedron to the opposite vertex. How many lines are there in the set D? Identify the stabilizer in G of one of the lines in D.

6D Groups

Let $S(X)$ denote the group of permutations of a finite set X. Show that every permutation $\sigma \in S(X)$ can be written as a product of disjoint cycles. Explain briefly why two permutations in $S(X)$ are conjugate if and only if, when they are written as the product of disjoint cycles, they have the same number of cycles of length n for each possible value of n.

Let $\ell(\sigma)$ denote the number of disjoint cycles, including 1-cycles, required when σ is written as a product of disjoint cycles. Let τ be a transposition in $S(X)$ and σ any permutation in $S(X)$. Prove that $\ell(\tau\sigma) = \ell(\sigma) \pm 1$.

7D Groups

Define the *cross-ratio* $[a_0, a_1, a_2, z]$ of four points a_0, a_1, a_2, z in $\mathbb{C} \cup \{\infty\}$, with a_0, a_1, a_2 distinct.

Let a_0, a_1, a_2 be three distinct points. Show that, for every value $w \in \mathbb{C} \cup \{\infty\}$, there is a unique point $z \in \mathbb{C} \cup \{\infty\}$ with $[a_0, a_1, a_2, z] = w$. Let S be the set of points z for which the cross-ratio $[a_0, a_1, a_2, z]$ is in $\mathbb{R} \cup \{\infty\}$. Show that S is either a circle or else a straight line together with ∞.

A map $J : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ satisfies

$$[a_0, a_1, a_2, J(z)] = \overline{[a_0, a_1, a_2, z]}$$

for each value of z. Show that this gives a well-defined map J with J^2 equal to the identity.

When the three points a_0, a_1, a_2 all lie on the real line, show that J must be the conjugation map $J : z \mapsto \overline{z}$. Deduce from this that, for any three distinct points a_0, a_1, a_2, the map J depends only on the circle (or straight line) through a_0, a_1, a_2 and not on their particular values.

8D Groups

What does it mean to say that a subgroup K of a group G is *normal*?

Let $\phi : G \to H$ be a group homomorphism. Is the kernel of ϕ always a subgroup of G? Is it always a normal subgroup? Is the image of ϕ always a subgroup of H? Is it always a normal subgroup? Justify your answers.

Let $\mathrm{SL}(2,\mathbb{Z})$ denote the set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Show that $\mathrm{SL}(2,\mathbb{Z})$ is a group under matrix multiplication. Similarly, when $\mathbb{Z}_2$ denotes the integers modulo 2, let $\mathrm{SL}(2,\mathbb{Z}_2)$ denote the set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}_2$ and $ad - bc = 1$. Show that $\mathrm{SL}(2,\mathbb{Z}_2)$ is also a group under matrix multiplication.

Let $f : \mathbb{Z} \to \mathbb{Z}_2$ send each integer to its residue modulo 2. Show that

$$\phi : \mathrm{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}_2)\ ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} f(a) & f(b) \\ f(c) & f(d) \end{pmatrix}$$

is a group homomorphism. Show that the image of ϕ is isomorphic to a permutation group.

9B Vector Calculus

Let S be a bounded region of $\mathbb{R}^2$ and ∂S be its boundary. Let u be the unique solution to Laplace's equation in S, subject to the boundary condition $u = f$ on ∂S, where f is a specified function. Let w be any smooth function with $w = f$ on ∂S. By writing $w = u + \delta$, or otherwise, show that

$$\int_S |\boldsymbol{\nabla} w|^2 \,\mathrm{d}A \geqslant \int_S |\boldsymbol{\nabla} u|^2 \,\mathrm{d}A\ . \qquad (*)$$

Let S be the unit disc in $\mathbb{R}^2$. By considering functions of the form $g(r)\cos\theta$ on both sides of $(*)$, where r and θ are polar coordinates, deduce that

$$\int_0^1 \left(r\left(\frac{\mathrm{d}g}{\mathrm{d}r}\right)^2 + \frac{g^2}{r} \right) \mathrm{d}r \geqslant 1$$

for any differentiable function $g(r)$ satisfying $g(1) = 1$ and for which the integral converges at $r = 0$.

$$\left[\boldsymbol{\nabla} f(r,\theta) = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta} \right), \qquad \nabla^2 f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}\,. \right]$$

10B Vector Calculus

Give a necessary condition for a given vector field $\mathbf{J}$ to be the curl of another vector field $\mathbf{B}$. Is the vector field $\mathbf{B}$ unique? If not, explain why not.

State Stokes' theorem and use it to evaluate the area integral

$$\int_S \left(y^2, z^2, x^2\right) \cdot \mathbf{dA} \,,$$

where S is the half of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

that lies in $z \geqslant 0$, and the area element $\mathbf{dA}$ points out of the ellipsoid.

11B Vector Calculus

A second-rank tensor $T(\mathbf{y})$ is defined by

$$T_{ij}(\mathbf{y}) = \int_S (y_i - x_i)(y_j - x_j)|\mathbf{y} - \mathbf{x}|^{2n-2}\, \mathrm{d}A(\mathbf{x})\,,$$

where $\mathbf{y}$ is a fixed vector with $|\mathbf{y}| = a$, $n > -1$, and the integration is over all points $\mathbf{x}$ lying on the surface S of the sphere of radius a, centred on the origin. Explain briefly why T might be expected to have the form

$$T_{ij} \;=\; \alpha\delta_{ij} + \beta y_i\, y_j\,,$$

where α and β are scalar constants.

Show that $\mathbf{y} \cdot (\mathbf{y} - \mathbf{x}) = a^2(1 - \cos\theta)$, where θ is the angle between $\mathbf{y}$ and $\mathbf{x}$, and find a similar expression for $|\mathbf{y} - \mathbf{x}|^2$. Using suitably chosen spherical polar coordinates, show that

$$y_i T_{ij}\, y_j \;=\; \frac{\pi a^2 (2a)^{2n+2}}{n+2}\,.$$

Hence, by evaluating another scalar integral, determine α and β, and find the value of n for which T is isotropic.

12B Vector Calculus

State the divergence theorem for a vector field $\mathbf{u}(\mathbf{x})$ in a region V of $\mathbb{R}^3$ bounded by a smooth surface S.

Let $f(x, y, z)$ be a homogeneous function of degree n, that is, $f(kx, ky, kz) = k^n f(x, y, z)$ for any real number k. By differentiating with respect to k, show that

$$\mathbf{x} \cdot \boldsymbol{\nabla} f = nf \ .$$

Deduce that

$$\int_V f \, \mathrm{d}V = \frac{1}{n+3} \int_S f \, \mathbf{x} \cdot \mathbf{dA} \ . \tag{$\dagger$}$$

Let V be the cone $0 \leqslant z \leqslant \alpha$, $\alpha\sqrt{x^2 + y^2} \leqslant z$, where α is a positive constant. Verify that ($\dagger$) holds for the case $f = z^4 + \alpha^4(x^2 + y^2)^2$.

END OF PAPER