UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Tuesday 5th June 2007 1.30 pm to 4.30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A*** *and* ***D*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Algebra and Geometry

Prove that every permutation of $\{1, \ldots, n\}$ may be expressed as a product of disjoint cycles.

Let $\sigma = (1234)$ and let $\tau = (345)(678)$. Write $\sigma\tau$ as a product of disjoint cycles. What is the order of $\sigma\tau$?

2D Algebra and Geometry

What does it mean to say that groups G and H are *isomorphic*?

Prove that no two of C_8, $C_4 \times C_2$ and $C_2 \times C_2 \times C_2$ are isomorphic. [Here C_n denotes the cyclic group of order n.]

Give, with justification, a group of order 8 that is not isomorphic to any of those three groups.

3A Vector Calculus

(i) Give definitions for the unit tangent vector $\hat{\mathbf{T}}$ and the curvature κ of a parametrised curve $\mathbf{x}(t)$ in $\mathbb{R}^3$. Calculate $\hat{\mathbf{T}}$ and κ for the circular helix

$$\mathbf{x}(t) = (a\cos t\,,\ a\sin t\,,\ bt)\,,$$

where a and b are constants.

(ii) Find the normal vector and the equation of the tangent plane to the surface S in $\mathbb{R}^3$ given by

$$z = x^2y^3 - y + 1$$

at the point $x = 1$, $y = 1$, $z = 1$.

4A Vector Calculus

By using suffix notation, prove the following identities for the vector fields $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{R}^3$:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})\,;$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B})\,.$$

SECTION II

5D Algebra and Geometry

Let x be an element of a finite group G. What is meant by the *order* of x? Prove that the order of x must divide the order of G. [*No version of Lagrange's theorem or the Orbit-Stabilizer theorem may be used without proof.*]

If G is a group of order n, and d is a divisor of n with $d < n$, is it always true that G must contain an element of order d? Justify your answer.

Prove that if m and n are coprime then the group $C_m \times C_n$ is cyclic.

If m and n are not coprime, can it happen that $C_m \times C_n$ is cyclic?

[Here C_n denotes the cyclic group of order n.]

6D Algebra and Geometry

What does it mean to say that a subgroup H of a group G is *normal*? Give, with justification, an example of a subgroup of a group that is normal, and also an example of a subgroup of a group that is not normal.

If H is a normal subgroup of G, explain carefully how to make the set of (left) cosets of H into a group.

Let H be a normal subgroup of a finite group G. Which of the following are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) If G is cyclic then H and G/H are cyclic.

(ii) If H and G/H are cyclic then G is cyclic.

(iii) If G is abelian then H and G/H are abelian.

(iv) If H and G/H are abelian then G is abelian.

7D Algebra and Geometry

Let A be a real symmetric $n \times n$ matrix. Prove that every eigenvalue of A is real, and that eigenvectors corresponding to distinct eigenvalues are orthogonal. Indicate clearly where in your argument you have used the fact that A is real.

What does it mean to say that a real $n \times n$ matrix P is *orthogonal*? Show that if P is orthogonal and A is as above then $P^{-1}AP$ is symmetric. If P is any real invertible matrix, must $P^{-1}AP$ be symmetric? Justify your answer.

Give, with justification, real 2×2 matrices B, C, D, E with the following properties:

(i) B has no real eigenvalues;

(ii) C is not diagonalisable over $\mathbb{C}$;

(iii) D is diagonalisable over $\mathbb{C}$, but not over $\mathbb{R}$;

(iv) E is diagonalisable over $\mathbb{R}$, but does not have an orthonormal basis of eigenvectors.

8D Algebra and Geometry

In the group of Möbius maps, what is the order of the Möbius map $z \mapsto \dfrac{1}{z}$? What is the order of the Möbius map $z \mapsto \dfrac{1}{1-z}$?

Prove that every Möbius map is conjugate either to a map of the form $z \mapsto \mu z$ (some $\mu \in \mathbb{C}$) or to the map $z \mapsto z+1$. Is $z \mapsto z+1$ conjugate to a map of the form $z \mapsto \mu z$?

Let f be a Möbius map of order n, for some positive integer n. Under the action on $\mathbb{C} \cup \{\infty\}$ of the group generated by f, what are the various sizes of the orbits? Justify your answer.

9A Vector Calculus

(i) Define what is meant by a conservative vector field. Given a vector field $\mathbf{A} = (A_1(x,y), A_2(x,y))$ and a function $\psi(x,y)$ defined in $\mathbb{R}^2$, show that, if $\psi\mathbf{A}$ is a conservative vector field, then

$$\psi\left(\frac{\partial A_1}{\partial y} - \frac{\partial A_2}{\partial x}\right) = A_2\frac{\partial \psi}{\partial x} - A_1\frac{\partial \psi}{\partial y}\,.$$

(ii) Given two functions $P(x,y)$ and $Q(x,y)$ defined in $\mathbb{R}^2$, prove Green's theorem,

$$\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy\,,$$

where C is a simple closed curve bounding a region R in $\mathbb{R}^2$.

Through an appropriate choice for P and Q, find an expression for the area of the region R, and apply this to evaluate the area of the ellipse bounded by the curve

$$x = a\cos\theta\,, \qquad y = b\sin\theta\,, \qquad 0 \le \theta \le 2\pi\,.$$

10A Vector Calculus

For a given charge distribution $\rho(x, y, z)$ and divergence-free current distribution $\mathbf{J}(x, y, z)$ (i.e. $\nabla \cdot \mathbf{J} = 0$) in $\mathbb{R}^3$, the electric and magnetic fields $\mathbf{E}(x, y, z)$ and $\mathbf{B}(x, y, z)$ satisfy the equations

$$\nabla \times \mathbf{E} = 0\,, \qquad \nabla \cdot \mathbf{B} = 0\,, \qquad \nabla \cdot \mathbf{E} = \rho\,, \qquad \nabla \times \mathbf{B} = \mathbf{J}\,.$$

The radiation flux vector $\mathbf{P}$ is defined by $\mathbf{P} = \mathbf{E} \times \mathbf{B}$.

For a closed surface S around a region V, show using Gauss' theorem that the flux of the vector $\mathbf{P}$ through S can be expressed as

$$\iint_S \mathbf{P} \cdot d\mathbf{S} = - \iiint_V \mathbf{E} \cdot \mathbf{J}\, dV\,. \qquad (*)$$

For electric and magnetic fields given by

$$\mathbf{E}(x, y, z) = (z,\ 0,\ x)\,, \qquad \mathbf{B}(x, y, z) = (0,\ -xy,\ xz)\,,$$

find the radiation flux through the quadrant of the unit spherical shell given by

$$x^2 + y^2 + z^2 = 1\,, \quad \text{with} \quad 0 \le x \le 1\,, \quad 0 \le y \le 1\,, \quad -1 \le z \le 1\,.$$

[*If you use* (∗)*, note that an open surface has been specified.*]

11A Vector Calculus

The function $\phi(x, y, z)$ satisfies $\nabla^2\phi = 0$ in V and $\phi = 0$ on S, where V is a region of $\mathbb{R}^3$ which is bounded by the surface S. Prove that $\phi = 0$ everywhere in V.

Deduce that there is at most one function $\psi(x, y, z)$ satisfying $\nabla^2\psi = \rho$ in V and $\psi = f$ on S, where $\rho(x, y, z)$ and $f(x, y, z)$ are given functions.

Given that the function $\psi = \psi(r)$ depends only on the radial coordinate $r = |\mathbf{x}|$, use Cartesian coordinates to show that

$$\nabla\psi = \frac{1}{r}\frac{d\psi}{dr}\mathbf{x}\,, \qquad \nabla^2\psi = \frac{1}{r}\frac{d^2(r\psi)}{dr^2}\,.$$

Find the general solution in this radial case for $\nabla^2\psi = c$ where c is a constant.

Find solutions $\psi(r)$ for a solid sphere of radius $r = 2$ with a central cavity of radius $r = 1$ in the following three regions:

(i) $0 \leqslant r \leqslant 1$ where $\nabla^2\psi = 0$ and $\psi(1) = 1$ and ψ bounded as $r \to 0$;

(ii) $1 \leqslant r \leqslant 2$ where $\nabla^2\psi = 1$ and $\psi(1) = \psi(2) = 1$;

(iii) $r \geqslant 2$ where $\nabla^2\psi = 0$ and $\psi(2) = 1$ and $\psi \to 0$ as $r \to \infty$.

12A Vector Calculus

Show that any second rank Cartesian tensor P_{ij} in $\mathbb{R}^3$ can be written as a sum of a symmetric tensor and an antisymmetric tensor. Further, show that P_{ij} can be decomposed into the following terms

$$P_{ij} = P\delta_{ij} + S_{ij} + \epsilon_{ijk}A_k\,, \qquad (\dagger)$$

where S_{ij} is symmetric and traceless. Give expressions for P, S_{ij} and A_k explicitly in terms of P_{ij}.

For an isotropic material, the stress P_{ij} can be related to the strain T_{ij} through the stress–strain relation, $P_{ij} = c_{ijkl}\,T_{kl}$, where the elasticity tensor is given by

$$c_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

and α, β and γ are scalars. As in (†), the strain T_{ij} can be decomposed into its trace T, a symmetric traceless tensor W_{ij} and a vector V_k. Use the stress–strain relation to express each of T, W_{ij} and V_k in terms of P, S_{ij} and A_k.

Hence, or otherwise, show that if T_{ij} is symmetric then so is P_{ij}. Show also that the stress-strain relation can be written in the form

$$P_{ij} = \lambda\,\delta_{ij}T_{kk} + \mu\,T_{ij}\,,$$

where μ and λ are scalars.

END OF PAPER