UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday 2 June 2005 9 to 12

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II, no more than* ***three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a gold cover sheet to each bundle; write the code letter in the box marked 'EXAMINER LETTER' on the cover sheet.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIRMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Algebra and Geometry

Convert the following expressions from suffix notation (assuming the summation convention in three dimensions) into standard notation using vectors and/or matrices, where possible, identifying the one expression that is incorrectly formed:

(i) δ_{ij},

(ii) $\delta_{ii}\,\delta_{ij}$,

(iii) $\delta_{ll}\,a_i\,b_j\,C_{ij}\,d_k - C_{ik}\,d_i$,

(iv) $\epsilon_{ijk}\,a_k\,b_j$,

(v) $\epsilon_{ijk}\,a_j\,a_k$.

Write the vector triple product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ in suffix notation and derive an equivalent expression that utilises scalar products. Express the result both in suffix notation and in standard vector notation. Hence or otherwise determine $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ when $\mathbf{a}$ and $\mathbf{b}$ are orthogonal and $\mathbf{c}=\mathbf{a}+\mathbf{b}+\mathbf{a}\times\mathbf{b}$.

2B Algebra and Geometry

Let $\mathbf{n}\in\mathbb{R}^3$ be a unit vector. Consider the operation

$$\mathbf{x}\mapsto\mathbf{n}\times\mathbf{x}\,.$$

Write this in matrix form, i.e., find a 3×3 matrix $\mathbf{A}$ such that $\mathbf{A}\mathbf{x}=\mathbf{n}\times\mathbf{x}$ for all $\mathbf{x}$, and compute the eigenvalues of $\mathbf{A}$. In the case when $\mathbf{n}=(0,0,1)$, compute $\mathbf{A}^2$ and its eigenvalues and eigenvectors.

3F Analysis

Define the *supremum* or *least upper bound* of a non-empty set of real numbers.

Let A denote a non-empty set of real numbers which has a supremum but no maximum. Show that for every $\epsilon>0$ there are infinitely many elements of A contained in the open interval

$$(\sup A-\epsilon\,,\ \sup A).$$

Give an example of a non-empty set of real numbers which has a supremum *and* maximum and for which the above conclusion does not hold.

4D Analysis

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series in the complex plane with radius of convergence R. Show that $|a_n z^n|$ is unbounded in n for any z with $|z| > R$. State clearly any results on absolute convergence that are used.

For every $R \in [0, \infty]$, show that there exists a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence R.

SECTION II

5C Algebra and Geometry

Give the real and imaginary parts of each of the following functions of $z = x + iy$, with x, y real,

(i) e^z,

(ii) $\cos z$,

(iii) $\log z$,

(iv) $\dfrac{1}{z} + \dfrac{1}{\bar{z}}$,

(v) $z^3 + 3z^2\bar{z} + 3z\bar{z}^2 + \bar{z}^3 - \bar{z}$,

where $\bar{z}$ is the complex conjugate of z.

An ant lives in the complex region R given by $|z-1| \leq 1$. Food is found at z such that

$$(\log z)^2 = -\frac{\pi^2}{16}.$$

Drink is found at z such that

$$\frac{z + \frac{1}{2}\bar{z}}{\left(z - \frac{1}{2}\bar{z}\right)^2} = 3, \quad z \neq 0.$$

Identify the places within R where the ant will find the food or drink.

6B Algebra and Geometry

Let $\mathbf{A}$ be a real 3×3 matrix. Define the rank of $\mathbf{A}$. Describe the space of solutions of the equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,, \qquad (\dagger)$$

organizing your discussion with reference to the rank of $\mathbf{A}$.

Write down the equation of the tangent plane at $(0,1,1)$ on the sphere $x_1^2 + x_2^2 + x_3^2 = 2$ and the equation of a general line in $\mathbb{R}^3$ passing through the origin $(0,0,0)$.

Express the problem of finding points on the intersection of the tangent plane and the line in the form $(\dagger)$. Find, and give geometrical interpretations of, the solutions.

7A Algebra and Geometry

Consider two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$. Show that $\mathbf{a}$ may be written as the sum of two vectors: one parallel (or anti-parallel) to $\mathbf{b}$ and the other perpendicular to $\mathbf{b}$. By setting the former equal to $\cos\theta|\mathbf{a}|\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is a unit vector along $\mathbf{b}$, show that

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\,.$$

Explain why this is a sensible definition of the angle θ between $\mathbf{a}$ and $\mathbf{b}$.

Consider the 2^n vertices of a cube of side 2 in $\mathbb{R}^n$, centered on the origin. Each vertex is joined by a straight line through the origin to another vertex: the lines are the 2^{n-1} diagonals of the cube. Show that no two diagonals can be perpendicular if n is odd.

For $n = 4$, what is the greatest number of mutually perpendicular diagonals? List all the possible angles between the diagonals.

8A Algebra and Geometry

Given a non-zero vector v_i, any 3×3 symmetric matrix T_{ij} can be expressed as

$$T_{ij} = A\delta_{ij} + Bv_iv_j + (C_iv_j + C_jv_i) + D_{ij}$$

for some numbers A and B, some vector C_i and a symmetric matrix D_{ij}, where

$$C_iv_i = 0, \qquad D_{ii} = 0, \qquad D_{ij}v_j = 0\,,$$

and the summation convention is implicit.

Show that the above statement is true by finding A, B, C_i and D_{ij} explicitly in terms of T_{ij} and v_j, or otherwise. Explain why A, B, C_i and D_{ij} together provide a space of the correct dimension to parameterise an arbitrary symmetric 3×3 matrix T_{ij}.

9F Analysis

Examine each of the following series and determine whether or not they converge. Give reasons in each case.

(*i*)
$$\sum_{n=1}^{\infty} \frac{1}{n^2} \, ,$$

(*ii*)
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + (-1)^{n+1}\, 2n + 1} \, ,$$

(*iii*)
$$\sum_{n=1}^{\infty} \frac{n^3 + (-1)^n\, 8n^2 + 1}{n^4 + (-1)^{n+1}\, n^2} \, ,$$

(*iv*)
$$\sum_{n=1}^{\infty} \frac{n^3}{e^{e^n}} \, .$$

10D Analysis

Explain what it means for a bounded function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable.

Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function. Show that for each $x \in (0, \infty)$, there exists a unique point $g(x) \in (0, x)$ such that

$$\frac{1}{x} \int_0^x f(t)\, dt = f(g(x)).$$

Find $g(x)$ if $f(x) = e^{-x}$.

Suppose now that f is differentiable and $f'(x) < 0$ for all $x \in (0, \infty)$. Prove that g is differentiable at all $x \in (0, \infty)$ and $g'(x) > 0$ for all $x \in (0, \infty)$, stating clearly any results on the inverse of f you use.

11E Analysis

Prove that if f is a continuous function on the interval $[a, b]$ with $f(a) < 0 < f(b)$ then $f(c) = 0$ for some $c \in (a, b)$.

Let g be a continuous function on $[0, 1]$ satisfying $g(0) = g(1)$. By considering the function $f(x) = g(x + \frac{1}{2}) - g(x)$ on $[0, \frac{1}{2}]$, show that $g(c + \frac{1}{2}) = g(c)$ for some $c \in [0, \frac{1}{2}]$. Show, more generally, that for any positive integer n there exists a point $c_n \in [0, \frac{n-1}{n}]$ for which $g(c_n + \frac{1}{n}) = g(c_n)$.

12E Analysis

State and prove Rolle's Theorem.

Prove that if the real polynomial p of degree n has all its roots real (though not necessarily distinct), then so does its derivative p'. Give an example of a cubic polynomial p for which the converse fails.

END OF PAPER