UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday, 28 May, 2010 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the left hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

Solutions of the equation

$$\left(1 - \frac{1}{x}\right)\frac{d^2 y}{d\,x^2} + \left(\frac{2}{x} - \frac{1}{x^2}\right)\frac{dy}{dx} - \frac{l(l+1)}{x^2}\,y \;=\; -\lambda\, y\,, \qquad (*)$$

with $l = 1,\, 2,\, \ldots$ behave for small but positive $(x-1)$ like

$$c_1 + c_2\, \ln(x-1)\,,$$

where c_1 and c_2 are constants. An eigenvalue problem is defined by the condition that real valued solutions on the interval $1 \leqslant x \leqslant R$ are subject to the boundary conditions that $y(R) = 0$ and $y(x)$ and dy/dx are bounded as x tends to 1 from above.

[6] (i) Show that the equation (*) may be cast into self-adjoint form.

(ii) Give the self-adjoint operator and verify, subject to the boundary conditions, that it
[4] is indeed self-adjoint.

[5] (iii) Show that the eigenvalues λ must be real and greater than zero.

(iv) Show explicitly, using the boundary conditions, that eigenfunctions y_i and y_j with different eigenvalues $\lambda_i \neq \lambda_j$ are orthogonal with respect to a suitably weighted inner
[5] product.

2A

Show that Laplace's equation in plane polar coordinates r, θ,

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0$$

admits solutions of the form

$$\Phi = a_0 + b_0 \ln(r) + \sum_{n=1}^{n=\infty}\left(a_n r^n + \frac{b_n}{r^n}\right)\Big(A_n \cos(n\theta) + B_n \sin(n\theta)\Big).$$

[4]

Find a solution $\Phi(r, \theta)$ which is bounded inside the disc $r \leqslant R$ and such that

$$\Phi(R,\theta) = \begin{cases} 0, & \frac{\pi}{2} \leqslant \theta \leqslant \pi, \\ C, & -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \\ 0, & -\pi \leqslant \theta \leqslant -\frac{\pi}{2}, \end{cases}$$

[6] where C is a constant.

Show that your solution is unique subject to the stated boundary conditions by supposing to the contrary the existence of another solution $\tilde{\Phi}$ satisfying the same boundary conditions, and applying the divergence theorem to

$$\int_D (\tilde{\Phi} - \Phi)\ \nabla^2(\tilde{\Phi} - \Phi)\, dx\, dy\,,$$

[4] where the domain D is the disc $r \leqslant R$.

Construct a solution bounded outside the disc, i.e. for $r \geqslant R$, with the same
[2] boundary data on the circle $r = R$ and which tends to a constant at infinity.

Show that the constant is not freely specifiable but must take a certain value which
[2] should be specified.

Show further, using the divergence theorem, that there is no other bounded solution
[2] taking that value.

3A

If

$$G_F(\mathbf{x},\,\mathbf{y}) \;=\; \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\;e^{i\omega|\mathbf{x}-\mathbf{y}|}\,,$$

and if $\Psi(\mathbf{x})$ satisfies the equation

$$\frac{\partial^2\Psi}{\partial x_1^2}+\frac{\partial^2\Psi}{\partial x_2^2}+\frac{\partial^2\Psi}{\partial x_3^2} \;=\; -\omega^2\,\Psi \qquad (*)$$

in a volume V, by applying Green's identity show that

$$\int_{\partial V}\Big(\boldsymbol{\nabla}G_F(\mathbf{x},\,\mathbf{y})\Psi(\mathbf{y})-G_F(\mathbf{x},\,\mathbf{y})\;\boldsymbol{\nabla}\Psi(\mathbf{y})\Big)\cdot d\mathbf{S}\,(\mathbf{y}) \;\left.\begin{aligned} &= -\Psi(\mathbf{x})\,, && \mathbf{x}\in V, \\ &= 0\,, && \mathbf{x}\notin V\,. \end{aligned}\right\} \qquad (**)$$

where ∂V is a closed surface with **outward** unit normal $\mathbf{n}$ which encloses the volume V. For surfaces we write the surface area element $d\mathbf{S}=\mathbf{n}dS$, and in (**) the gradient
[4] operator $\boldsymbol{\nabla}=\mathbf{i}\,\dfrac{\partial}{\partial y_1}+\mathbf{j}\,\dfrac{\partial}{\partial y_2}+\mathbf{k}\,\dfrac{\partial}{\partial y_3}$.

Now assume that Ψ satisfies (*) in the half-space $\{x_3>0\}$. By applying Green's identity, and taking into account the integral over a large hemisphere in the half-space $\{x_3>0\}$, show that if

$$\left.\begin{aligned} \lim_{|\mathbf{y}|\to\infty}\,|\mathbf{y}\,\Psi(\mathbf{y})| &\leqslant \infty\,, \\ \lim_{|\mathbf{y}|\to\infty}\big(\mathbf{y}\cdot\;\boldsymbol{\nabla}-i\omega|\mathbf{y}|\big)\,\Psi(\mathbf{y}) &= 0\,, \end{aligned}\right\} \qquad (\dagger)$$

then

$$\int_{y_3=0}\Big(\Psi(\mathbf{y})\boldsymbol{\nabla}G_F(\mathbf{x},\,\mathbf{y})-G_F(\mathbf{x},\,\mathbf{y})\boldsymbol{\nabla}\Psi(\mathbf{y})\Big)\cdot d\mathbf{S}(\mathbf{y}) = -\Psi(\mathbf{x})\,, \quad \text{for } \mathbf{x} \text{ such that } x_3>0\,,$$

[6] $$= \;0\,, \quad \text{for } \mathbf{x} \text{ such that } x_3<0\,.$$

Hence show that for $x_3>0$,

$$\Psi(\mathbf{x}) \;=\; -\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial\Psi}{\partial x_3}\,(y_1,\,y_2,0)\;\frac{e^{\,i\omega R}}{R}\;dy_1\,dy_2\,,$$

where $R=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+x_3^2}$. [*You may find it useful to consider also the*
[6] *function* $G_F((x_1,x_2,-x_3),\mathbf{y})$.]

Hence show that if $\Psi(\mathbf{x})$ satisfies the boundary conditions (†) together with

$$\frac{\partial\Psi}{\partial x_3}\,(x_1,\,x_2,\,0)=0\,,$$

[4] then $\Psi(\mathbf{x})=0$ for all $x_3>0$.

4A

If a, b, c are real positive constants such that $a^2 > b^2 + c^2 > 0$, find the poles, z_1, z_2 of the analytic function

[4]
$$f(z) = \frac{1}{az + \frac{1}{2}b(z^2+1) + \frac{1}{2i}c(z^2-1)}.$$

Show that

$$|z_1 z_2| = 1,$$

[3] and hence that one pole lies inside and one outside the unit circle.

[3] Are the poles simple?

Hence show, using the contour $|z| = 1$ and Cauchy's Theorem, how one may evaluate the integral

[5]
$$I = \int_{-\pi}^{\pi} \frac{d\theta}{a + b\cos\theta + c\sin\theta}.$$

[5] Give the value of I.

5A

(i) Find an ordinary differential equation satisfied by the Fourier transform

$$\tilde{\theta}(k,t) = \int_{-\infty}^{\infty} e^{-ikx}\,\theta(x,t)\,dx$$

of a solution $\theta(x,t)$ of the heat equation

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2},$$

[2] on the interval $-\infty < x < \infty$ for $t \geqslant 0$.

(ii) Give an expression for the solution $\theta(x,t)$ in terms of the Fourier transform of the
[2] initial distribution of temperature $\tilde{\theta}(k,0)$.

(iii) Use the convolution theorem to express $\theta(x,t)$ as a convolution

$$\theta(x,t) = \int_{-\infty}^{\infty} \theta(y,0)\,G(x-y,\,t)\,dy\,,$$

[5] giving an explicit form for $G(x-y,\,t)$.

(iv) Suppose that the initial distribution is of Gaussian form:

$$\theta(x,0) = A_0\,e^{-a_0(x-x_0)^2}\,, \qquad (*)$$

where x_0, A_0 and $a_0 > 0$ are constants. Show that $\theta(x,t)$ is of Gaussian form and give an
[6] explicit formula for it.

Find an expression for $\theta(x,t)$ in terms of the error function $\mathrm{erf}(x)$ defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,du$$

in the case that

[5] $$\theta(x,0) = \begin{cases} = & 0\,, & x < a\,, \\ = & B\,, & a < x < b\,, \\ = & 0\,, & x > b\,. \end{cases} \qquad (\dagger)$$

Hence show that at late times, in the interval $a < x < b$,

$$\theta(x,t) \approx \frac{B(b-a)}{\sqrt{4\pi t}}\,.$$

Show in both cases (*) and (†) that at late times the heat has spread out over a distance
[2] which is $O(\sqrt{t})$.

[*You may assume that*

$$\int_{-\infty}^{\infty} e^{-(x+iy)^2}\,dx = \sqrt{\pi}$$

for x and y real.]

6B Define the terms *tensor* and *isotropic tensor*. Show that ϵ_{ijk}, the completely anti-symmetric tensor with $\epsilon_{123} = 1$, is isotropic. Let A be a rank two tensor with matrix entries A_{ij}. Using the formula

$$\det A \, \epsilon_{ijk} = \epsilon_{lmn} A_{il} A_{jm} A_{kn} \ ,$$

deduce that

$$\det A = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} A_{il} A_{jm} A_{kn} \ ,$$

and hence show that the determinant is a scalar. Show also that the inverse matrix A^{-1} has entries given by the formula

$$A^{-1}_{ba} = \frac{1}{2 \det A} \epsilon_{ajk} \epsilon_{bmn} A_{jm} A_{kn} \ .$$

[6]

Prove that the partial derivative $\frac{\partial \det A}{\partial A_{ab}}$ is given by

$$\frac{\partial \det A}{\partial A_{ab}} = \det A \, (A^{-1})^T_{ab} \ ,$$

[8] where ^{T} denotes matrix transpose.

Consider the case that $A_{ij}(t, \mathbf{x})$ arises as the Jacobian matrix of a smooth time-dependent transformation $x_i \to y_i = \Phi_i(t, \mathbf{x})$, i.e.

$$A_{ij}(t, \mathbf{x}) = \frac{\partial y_j}{\partial x_i} = \frac{\partial \Phi_j}{\partial x_i}(t, \mathbf{x}) \ ,$$

and assume that $\Phi_i(t, \mathbf{x}) = x_i + t u_i(\mathbf{x}) + O(t^2)$ for small t. By considering $\frac{d}{dt} \det A(t, \mathbf{x})$
[6] show that $\det A(t, \mathbf{x}) = 1 + O(t^2)$ for small t if $\mathrm{div}\, \mathbf{u} = \boldsymbol{\nabla} \cdot \mathbf{u} = 0$.

[*The summation convention is assumed throughout this question.*]

7B A mechanical system with N degrees of freedom $(q_1, \ldots q_N)$ described by a Lagrangian of the form

$$L = \frac{1}{2}\sum_{ij} T_{ij}\dot{q}_i\dot{q}_j - V(q_1, \ldots q_N)$$

where T_{ij} is a constant symmetric positive definite matrix, is subject to small oscillations about an equilibrium point. Define the *normal modes* and *normal frequencies.* State and derive the orthogonality relation for the normal modes. [3]

Consider three point masses of equal mass m, situated at points $\mathbf{x}_1 = (X_1, Y_1), \mathbf{x}_2 = (X_2, Y_2)$ and $\mathbf{x}_3 = (X_3, Y_3)$ in the plane, and connected by springs of equal unstretched length $l = \sqrt{3}$ and spring constant k. Write down the potential and kinetic energies and show that they are unchanged by an overall translation and by an overall rigid rotation of the system. Show that configurations in which the three masses lie at the vertices of equilateral triangles whose sides have length $l = \sqrt{3}$ are equilibrium points for the system. [1]

Write down the potential energy V for the system when the masses are located at $(X_1, Y_1) = (q_1, 1 + q_2)$, $(X_2, Y_2) = (\frac{\sqrt{3}}{2} + q_3, -\frac{1}{2} + q_4)$ and $(X_3, Y_3) = (-\frac{\sqrt{3}}{2} + q_5, -\frac{1}{2} + q_6)$. Show that for small $(q_1, \ldots q_6)$ the potential energy V can be expanded to quadratic order as [3]

$$\begin{aligned} V &= \frac{1}{2}k(q_5 - q_3)^2 + \frac{k}{2}\Big[\frac{1}{2}(q_1 - q_5) + \frac{\sqrt{3}}{2}(q_2 - q_6)\Big]^2 \\ &\quad + \frac{k}{2}\Big[-\frac{1}{2}(q_1 - q_3) + \frac{\sqrt{3}}{2}(q_2 - q_4)\Big]^2 + \ldots \end{aligned}$$

Set up the problem for small oscillations around the configuration in which the masses are at the vertices of an equilateral triangle centred at the origin, with the first mass situated at $(0, 1)$ and the remaining two at $(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2})$. Show that the Lagrangian for this system takes the form [6]

$$L = \frac{m}{2}\sum_j \dot{q}_j^2 - \frac{1}{2}\sum_{ij} V_{ij}q_iq_j, \qquad \text{where}$$

$$V_{ij} = \frac{k}{4}\begin{pmatrix} 2 & 0 & -1 & \sqrt{3} & -1 & -\sqrt{3} \\ 0 & 6 & \sqrt{3} & -3 & -\sqrt{3} & -3 \\ -1 & \sqrt{3} & 5 & -\sqrt{3} & -4 & 0 \\ \sqrt{3} & -3 & -\sqrt{3} & 3 & 0 & 0 \\ -1 & -\sqrt{3} & -4 & 0 & 5 & \sqrt{3} \\ -\sqrt{3} & -3 & 0 & 0 & \sqrt{3} & 3 \end{pmatrix}.$$

Use the translations and rigid rotations that you wrote down above to show that there are three normal modes of zero frequency, giving them explicitly. [4]

Prove that there is a normal mode in which all of the masses move radially and find its frequency. [3]

8B Define the *order* of a finite group G and state Lagrange's theorem on the order of a subgroup K of G. [2]

Prove that every order four group is either the cyclic group C_4 or is the Vierergruppe V, i.e. the order four abelian group $\{I, a, b, c\}$ in which $a^2 = b^2 = c^2 = I$ and $ab = c$. [5]

Define a *homomorphism* $\Phi : G \mapsto H$ between finite groups G and H. [1]

Prove that K, the kernel of a homomorphism $\Phi : G \mapsto H$, is a normal subgroup of G. Assuming that the image of Φ contains all of H, prove that the quotient group G/K is isomorphic to H. [4]

Consider the multiplicative group Q which has elements $\pm 1, \pm i, \pm j, \pm k$, where 1 is the identity, $(-1)^2 = 1$ and i, j, k satisfy

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = k, \quad jk = i \quad \text{and} \quad ki = j.$$

Show that these relations imply that (-1) commutes with i, j, k and deduce that $N = \{\pm 1\}$ is a normal subgroup of Q. [4]

Obtain a homomorphism $\Phi : Q \mapsto V$ (where V is as defined above) whose kernel is N, and give a quotient group of Q which is isomorphic to V. [4]

9B

If G is an arbitrary finite group, define the *conjugacy classes* of G and show that each element lies in a unique conjugacy class. [4]

Show that if G is Abelian then each element lies in a conjugacy class consisting only of itself, i.e. each element forms its own conjugacy class. [2]

If G is an arbitrary, not necessarily Abelian, finite group, its *centre* Z is defined to be the set of elements which each form their own conjugacy class. Prove that Z is an Abelian subgroup of G. [4]

Describe the group D_4 of symmetry operations of the square, including a geometrical description of the action of each element and a 2×2 matrix form for each such action. Give an example, with justification, of a pair of elements which are not conjugate to each other. [6]

Find the centre of D_4. [4]

10B Define the terms *representation*, *invariant subspace*, *irreducible representation* and *faithful representation.* Define the *character* of a representation, and state the orthogonality relation for characters. [5]

Consider the group Σ_3 of all possible permutations of three objects with the group operation defined by composition of the permutations. Display the group multiplication table and identify the conjugacy classes. [3]

Define a faithful three-dimensional representation of Σ_3 in which each element is represented by one of the following matrices:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

[2]

Show that the one-dimensional subspace V_0 of vectors of the form (c, c, c), for arbitrary real c, is an invariant subspace of the representation. Show that the subspace $V_0^{\perp}$ consisting of real vectors (v_1, v_2, v_3) with $v_1 + v_2 + v_3 = 0$ is the space of real vectors orthogonal to V_0, and prove that $V_0^{\perp}$ is also an invariant subspace of the representation. Find the eigenvectors of A and show that A has no (real) eigenvectors in $V_0^{\perp}$. Hence show that $V_0^{\perp}$ determines a two-dimensional irreducible representation of Σ_3. Hence decompose the above three-dimensional faithful representation into irreducible representations. [8]

Prove that Σ_3 has two one-dimensional irreducible representations, and one two-dimensional irreducible representation.

[*You may use without proof that if n_α is the dimension of the α^{th} irreducible representation of a finite group G, then the order of G is given by $|G| = \sum_\alpha n_\alpha^2$.*] [2]

END OF PAPER