UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday 1 June 2007 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

*You may submit answers to no more than **six** questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

*Write on **one** side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

*Each question has a number and a letter (for example, **6A**).*

*Answers must be tied up in **separate** bundles, marked **A, B or C** according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

*A **separate** yellow master cover sheet listing all the questions attempted **must** also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1B Axisymmetric solutions $\Phi(r,\theta)$ of Laplace's equation satisfy

$$\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial \Phi}{\partial r}\big) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\big(\sin\theta \frac{\partial \Phi}{\partial \theta}\big) = 0 \ ,$$

where (r,θ,ϕ) are the standard spherical polar co-ordinates. Show, by using the method of separation of variables, that the solution to this equation can be written as

$$\Phi(r,\theta) = \sum_{n=0}^{\infty} \big(A_n r^n + B_n r^{-n-1}\big) P_n(\cos\theta)$$

for constants A_n, B_n. Derive the equation which the functions P_n must satisfy. [12]

A spherically symmetric shell of material lies between $a < r < b$. The temperature on the boundaries of the shell is

$$T(a,\theta) = T_0 + T_1 \cos\theta, \qquad T(b,\theta) = T_2 + T_3 \cos\theta \ ,$$

for constants T_0, T_1, T_2, T_3. Determine the steady-state axisymmetric temperature within the shell.

[8]

[You may assume that $P_0(\cos\theta) = 1$ and $P_1(\cos\theta) = \cos\theta$].

2B Prove that the Green's function

$$G(\mathbf{x};\mathbf{x}_0) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}_0|}$$

is the fundamental solution in three dimensions satisfying $G(\mathbf{x};\mathbf{x}_0) \to 0$ as $|\mathbf{x}| \to \infty$ and

$$\nabla^2 G(\mathbf{x};\mathbf{x}_0) = \delta(\mathbf{x}-\mathbf{x}_0)\ . \qquad (\star)$$

[4]

Using the method of images find the Green's function $G(\mathbf{x};\mathbf{x}_0)$ satisfying $(\star)$ for $\mathbf{x} \in V$ and (fixed) $\mathbf{x}_0 \in V$ when:

(i) V is the half space of $\mathbb{R}^3$ with $z > 0$, $G = 0$ on $z = 0$, and $G \to 0$ as $|\mathbf{x}| \to \infty$ for $\mathbf{x} \in V$.

(ii) V is the interior of the sphere $r < a$, and $G = 0$ on $r = a$.

[8]

A point charge e is placed at $\mathbf{x}_0 \in V$, where V is a hollow hemisphere of radius a:

$$V = \{(x,y,z) : x^2 + y^2 + z^2 < a^2 \text{ and } z > 0\}\ .$$

The boundary of V is earthed. Derive the electrostatic potential in V.

[8]

3B

Let $f(z) = u + iv$ for real u, v be an analytic function of $z = x + iy$ for real x, y. Prove that u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\ .$$

[8]

Determine $f(z)$ when $u(x,y) = e^x\big((x^2 - y^2)\cos y - 2xy\sin y\big)$. [8]

Determine the singularities (poles or branch points) of the following functions:

$$\text{(i)}\ \ f(z) = z^{\frac{1}{2}} \log\Big(\frac{z-1}{z+1}\Big)\ , \qquad \text{(ii)}\ \ f(z) = \frac{z}{z^2 + 4z + 5}\ .$$

[4]

[You may assume that $\int y^2 \cos y dy = y^2 \sin y - 2\sin y + 2y\cos y$ *and* $\int y \sin y dy = \sin y - y\cos y$*].*

4B

State the Residue Theorem. [4]

Let n be a positive integer with $n \geqslant 3$. Identify the poles of

$$f(z) = \frac{1}{1+z^n} \ .$$

By integrating $f(z)$ along a contour enclosing only one pole, prove that

$$\int_0^\infty \frac{1}{1+x^n}\ dx = \frac{\pi}{n\sin(\frac{\pi}{n})} \ .$$

Furthermore, compute the integral [8]

$$I = \int_0^\infty \frac{x^{\frac{n}{2}}}{1+x^n}\ dx \ .$$

[8]

5B

(i) Define the Laplace transform $\bar{f}(p)$ of a function $f(t)$ (where $f(t) = 0$ for $t < 0$). State the Bromwich inverse integral formula for the inverse Laplace transform, and describe how the Bromwich contour should be chosen in the integral.

Assuming that $f(t) \to f(0)$ as $t \to 0$ from above, and also that $f(t)e^{-pt} \to 0$ as $t \to +\infty$, show that the Laplace transform of $f'(t)$ is $p\bar{f}(p) - f(0)$. [8]

(ii) The function $S_n(t)$ is defined by

$$S_n(t) = \begin{cases} n & \text{if } 1 - \frac{1}{2n} < t < 1 + \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

where n is a positive integer. The function $f(t)$ satisfies $f(0) = 0$ and

$$f'(t) - f(t) = S_n(t) \qquad (\star)$$

for $t > 0$, with $f(t) = 0$ for $t < 0$. Using the Laplace transform, find $f(t)$ for $t < 1 - \frac{1}{2n}$ and $t > 1 + \frac{1}{2n}$. [12]

[For this question, Jordan's Lemma may be used without proof, provided it is stated carefully].

6C

Define a tensor T of rank two in $\mathbb{R}^3$, and demonstrate that every such tensor can be decomposed as

$$T_{ij} = Y\delta_{ij} + \Omega_{ij} + S_{ij},$$

where Ω_{ij} is antisymmetric, S_{ij} is symmetric and traceless and Y is a scalar. What are the numbers of independent components of Ω_{ij} and S_{ij}? [6]

Let A_i be a non-zero rank one tensor and let T_{ij} be a symmetric rank two tensor. Find scalars α, β, a rank one tensor B_i, and a symmetric traceless rank two tensor C_{ij} such that

$$T_{ij} = \alpha\delta_{ij} + \beta A_i A_j + (B_i A_j + B_j A_i) + C_{ij} \ ,$$

$$A_i B_i = 0, \qquad C_{ij} A_i = 0 \ .$$

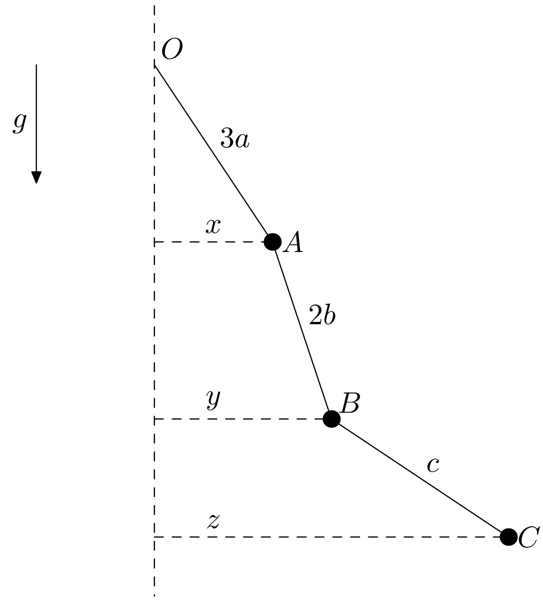
[14]

[The summation convention is assumed. It is not necessary to derive the transformation laws for Y, Ω_{ij}, S_{ij}, α, β, B_i and C_{ij}.]

7C

Write down a general Lagrangian of a system with n degrees of freedom undergoing small oscillations, and state the polynomial equation for the normal frequencies. [4]

Let A, B, C be three identical particles of unit mass, and let O be a fixed point. Assume that A is suspended from O by a string of length $3a$, B is suspended from A by a string of length $2b$ and C is suspended from B by a string of length c. The system moves in a plane containing A, B, C and O under the influence of gravity.



Let x, y, z denote the horizontal displacements of A, B, C from their equilibria. Show that the approximate Lagrangian for small oscillations is

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{g}{2}\Big(\frac{x^2}{a} + \frac{(x-y)^2}{b} + \frac{(y-z)^2}{c}\Big) \ ,$$

where g is the acceleration due to gravity. [8]

Show that the necessary and sufficient condition for the existence of a normal mode $y(t) = 0$ is

$$\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0 \ .$$

[8]

8C

State the Lagrange theorem relating the order of a group to orders of its subgroups. [2]

Let G be a group in which every element other than the identity has order 2. Show that this group is Abelian. Let a, b be distinct elements of G different from the identity element I. Show that $\{I, a, b, ab\}$ is a subgroup of G of order 4. [10]

Deduce that any group of order $2p$, where $p > 2$ is prime, must contain an element of order p. [8]

9C

Let G and H be two groups. Define the terms isomorphism, homomorphism and kernel, and show that if $\phi : G \longrightarrow H$ is a homomorphism then the kernel of ϕ is a normal subgroup of G. [6]

Show that matrices of the form

$$B = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} ,$$

where n is an integer, form a subgroup H of the multiplicative group of invertible 2×2 matrices. Is H cyclic? Does it have any proper subgroups? [8]

Demonstrate that H is isomorphic to the additive group of all integers. [6]

10C

Let G be a group, and let D be a map $D : G \to GL(2,\mathbb{R})$, where $GL(2,\mathbb{R})$ is the group of real 2×2 invertible matrices. What does it mean for D to be a representation of G? [4]

Let G be a cyclic subgroup of $GL(2,\mathbb{R})$ of order 4 given by

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} .$$

Suppose that real $x, y, \tilde{x}, \tilde{y}$ are related by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{for real } x, y, \text{ where} \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G .$$

Suppose in addition that $x, y, \tilde{x}, \tilde{y}$ satisfy the relation

$$\hat{a}\tilde{x}^2 + \hat{b}\tilde{y}^2 = ax^2 + by^2 \qquad (\star)$$

for real $\hat{a}, \hat{b}, a, b$. Find the real 2×2 matrix M, whose components do not depend on $\hat{a}, \hat{b}, a, b$, such that the relation $(\star)$ holds for all real x, y, and M satisfies

$$\begin{pmatrix} a \\ b \end{pmatrix} = M \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} .$$

[7]

Show that the map

$$A \to D(A) = M^{-1}$$

is a representation of G. Is this representation faithful? [9]

END OF PAPER