UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Friday 31 May, 2002 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6B****).*

Answers must be tied up in **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

1A A slow viscous flow in two dimensions is described by the biharmonic equation

$$\nabla^4\psi \equiv \nabla^2(\nabla^2\psi) = 0.$$

By looking for separable solutions of the form $\psi(r,\theta) = r^m \cos 2\theta$, where (r,θ) are plane polar coordinates, find a solution inside the circle $r = 1$ that satisfies $\psi = 0$ and $\frac{\partial\psi}{\partial r} = \cos 2\theta$ on $r = 1$ and $\psi = 0$ at $r = 0$. [13]

Find also a solution outside the circle $r = 1$ that satisfies the same boundary conditions on $r = 1$ and is such that $\frac{\partial\psi}{\partial r} \to 0$ as $r \to \infty$. [7]

[*In plane polar coordinates* (r,θ),

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}.]$$

2B Using the divergence theorem prove that

$$\int_V \left[\phi\left(\nabla^2\psi + k^2\psi\right) \; - \; \psi\left(\nabla^2\phi + k^2\phi\right)\right] dV \; = \; \int_S \left(\phi\nabla\psi - \psi\nabla\phi\right)\cdot \mathbf{dS}$$

where V is a simply connected volume enclosed by the surface S. [4]

Suppose that $\phi(\mathbf{x})$ satisfies the Helmholtz equation

$$\nabla^2\phi(\mathbf{x}) \; + \; k^2\phi(\mathbf{x}) \; = f(\mathbf{x}),$$

with $\phi(\mathbf{x}) = g(\mathbf{x})$ on S, and $G(\mathbf{x},\mathbf{x}')$ is the associated Green's function. Here k is a fixed real number. Show that

$$\phi(\mathbf{x}') \; = \; \int_V G(\mathbf{x},\mathbf{x}')f(\mathbf{x})dV \; + \; \int_S g(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n}dS$$

where n represents the direction normal to the surface S. [5]

If there are no boundary surfaces, you may assume that the Green's function is spherically symmetric around the point $\mathbf{x} = \mathbf{x}'$. Use this fact to find the general solution for G in infinite space for $\mathbf{x} \neq \mathbf{x}'$. [4]

By considering a neighborhood of $\mathbf{x} = \mathbf{x}'$ find the solution for G in terms of one undetermined parameter. [5]

Does the limit $k = 0$ reduce to the corresponding Green's function for the Laplace equation? [2]

[*The Laplacian in spherical coordinates can be written as:*

$$\nabla^2\psi(r,\theta,\phi) \; = \; \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

for an arbitrary function $\psi(r,\theta,\phi)$]

3A What is a branch point of a function $f(z)$? [2]

Find the branch points of $f(z) = (z^2 - 1)^{1/2}$ and describe how to make $f(z)$ single valued by making (a) one branch cut and (b) two branch cuts in the complex plane. [4]

For the shortest possible cut find the value of $f(z)$ in terms of x and y, where $z = x + iy$, on either side of the cut. [4]

Using integration by parts or otherwise show that

$$u(z) = \oint_C \frac{e^{z\zeta}}{(\zeta^2 - 1)^{1/2}}\, d\zeta$$

satisfies

$$zu'' + u' - zu = 0$$

when the closed contour C encloses both branch points of $f(z)$. [5]

By shrinking the contour on to the branch cut and taking care with integrating around the branch points find $u(0)$. [5]

4A In terms of a Laurent expansion for a function $f(z)$, in the complex plane, about $z = z_0$ define a pole, its order and its residue. [4]

Find the poles, and their orders and residues, of

$$f(z) = \frac{(z^2+1)^2}{z[a^2(z^2+1)^2 - (z^2-1)^2]},$$

when $0 < a \leq 1$. Pay particular attention to the special case $a = 1$. [9]

By integrating $f(z)$ around the unit circle and applying the calculus of residues show that

$$I = \int_0^{2\pi} \frac{1}{a^2 + \tan^2\theta} d\theta = \frac{2\pi}{a(1+a)}.$$

[7]

5A Define the Laplace transform $F(p)$ of a function $f(t)$ and write down the Bromwich inversion formula for $f(t)$ in terms of $F(p)$ clearly specifying the path of integration. [4]

Given a function $h(\lambda)$, define the integral transform

$$H(s) = \int_0^1 \lambda^{s-1} h(\lambda)\, d\lambda.$$

By considering $\lambda = e^{-t}$ show that the inverse of this integral transform is

$$h(\lambda) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \lambda^{-s} H(s)\, ds.$$

for a suitable γ. [6]

Use this formula to determine $h(\lambda)$ for $0 < \lambda < 1$ when

$$\text{(a)} \quad H(s) = \frac{1}{2s-1}$$

and

$$\text{(b)} \quad H(s) = \frac{1}{2^{s+1}(s+1)}.$$

[10]

6C (a) Write down the transformation law for a rank n tensor. Use this to define an *isotropic* tensor. [4]

(b) Show that any second rank tensor P_{ij} can be written as a sum of a symmetric tensor and an antisymmetric tensor. Hence, or otherwise, show that P_{ij} can be decomposed into the following terms

$$P_{ij} = P\,\delta_{ij} + S_{ij} + \epsilon_{ijk} A_k\,, \qquad (\dagger)$$

where S_{ij} is symmetric and traceless, $S_{ii} = 0$. Give the tensors P, S_{ij} and A_k explicitly in terms of P_{ij}. [8]

(c) The stress-strain equation,

$$P_{ij} = c_{ijk\ell}\, p_{k\ell}\,,$$

relates the stress P_{ij} on a material to the resulting strain p_{ij} (both second rank tensors) through the elasticity tensor $c_{ijk\ell}$ (fourth rank). For an isotropic material, use the stress-strain equation to express the decomposed components of P_{ij} (i.e. P, S_{ij} and A_k from (†)) in terms of the decomposed components of p_{ij} (denoted as p, s_{ij}, and a_k). Hence, show that if the strain p_{ij} is symmetric then so is the stress P_{ij}. [8]

[*You may assume that the most general fourth rank isotropic tensor is*

$$c_{ijk\ell} = \lambda \delta_{ij}\delta_{k\ell} + \mu \delta_{ik}\delta_{j\ell} + \nu \delta_{i\ell}\delta_{jk}\,,$$

where λ, μ and ν are scalars. The summation convention is assumed throughout.]

7B A system is subject to small oscillations. Define: *normal modes, normal frequencies and normal coordinates.* [4]

Consider three point masses m, at the vertices of an equilateral triangle of unit size, connected by springs of constant k. Take the origin of coordinates at the centre of the triangle with the three equilibrium points corresponding to $(-\frac{\sqrt{3}}{2},-\frac{1}{2}),(\frac{\sqrt{3}}{2},-\frac{1}{2})$, and $(0,1)$ for the first, second and third mass respectively. Each of the three masses is displaced by (x_i, y_i) , $i = 1, 2, 3$ with respect to the equilibrium points.

(i) Write down the potential and kinetic energies and then derive the equations of motion for each object. [5]

(ii) Show that the modes

$$(x_1, x_2, x_3; y_1, y_2, y_3) = (1, 1, 1; 0, 0, 0),\ (0, 0, 0; 1, 1, 1),\ (-1, -1, 2; \sqrt{3}, -\sqrt{3}, 0)$$

correspond to zero frequency oscillations and interpret the result. [5]

(iii) Find the remaining (orthogonal) normal modes of oscillation and their corresponding frequencies. [6]

[*There is no need to solve the characteristic equation to find the eigenvalues from the eigenvalue equation. It may be useful to first find three vectors orthogonal to those with zero frequencies and write the eigenvectors as linear combinations of them.*]

8B Given a finite group G of order $|G|$ and a subgroup H of order $|H|$, define the term *left coset of H in G*. State Lagrange's theorem relating the order of a group and its subgroups. [4]

Working with integers modulo 5 (for instance $2 = 7$ *mod* 5) and using arithmetic modulo 5 (so that $1 + 4$ *mod* $5 = 0$ for example), prove that the set of matrices of the form:

$$\mathbf{A} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

where here a, b, c are integers and $a \neq 0, c \neq 0$ form a finite group G under matrix multiplication. Show that G has 80 elements. [5]

Show that the subset given by $b = a - c$ defines an Abelian subgroup H. Find the order of H and verify Lagrange's theorem. [4]

How many distinct left cosets of H are in G? [2]

Find all the elements of G whose square is the 2×2 identity matrix and use Lagrange's theorem to prove that this subset cannot be a subgroup of G. [5]

9B For an arbitrary group G define what is meant by a *conjugacy class* of G. [4]

Prove that no element of G can be in two different conjugacy classes and therefore that each element belongs to a unique conjugacy class. [4]

If G is Abelian show that each element of G forms a class by itself. [4]

The *centre* Z of the group G is defined as the set of elements of G which form a conjugacy class by themselves. Prove that Z is an Abelian subgroup of G [6]

Find the centre of D_4, the symmetry group of the square. [2]

10B Define the terms *representation* and *irreducible representation* of a group G. [3]

Consider the set Σ_3 of all possible permutations of three objects (a, b, c) with an operation defined as the successive application of the permutations. A 3×3 matrix representation for each element of the group is given by the following set of matrices,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Construct the multiplication table for this group based on the corresponding matrix multiplication or otherwise. What are the conjugacy classes? [5]

From the knowledge of the order of the group, show that this is not an irreducible representation of Σ_3 and find the dimensions of all the irreducible representations of Σ_3. [5]

Compute the character of each conjugacy class in the 3×3 matrix representation defined above. Then construct the table of characters for the irreducible representations, without constructing the representations explicitly. Decompose the 3×3 matrix representation above in terms of the irreducible representations of the group of dimension 2 and 1. [7]

[*The orthogonality relation for characters may be used as well as the relation* $\sum_\alpha n_\alpha^2 = |G|$ *where* n_α *are the dimensions of the irreducible representations.*]