UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday, 31 May, 2011 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the left hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

Define the curl $\boldsymbol{\nabla} \times \mathbf{v}$ and the divergence $\boldsymbol{\nabla}.\mathbf{v}$ of a vector field $\mathbf{v}$ in Cartesian coordinates. Show that

$$\boldsymbol{\nabla} \times (\mathbf{v} \times [\boldsymbol{\nabla} \times \mathbf{v}]) = ([\boldsymbol{\nabla} \times \mathbf{v}].\boldsymbol{\nabla})\,\mathbf{v} - (\mathbf{v}.\boldsymbol{\nabla})\,(\boldsymbol{\nabla} \times \mathbf{v}) - (\boldsymbol{\nabla} \times \mathbf{v})(\boldsymbol{\nabla}.\mathbf{v})\,.$$

[6]

Define the scale factors (or metric coefficients) h_i $(i = 1, 2, 3)$ for a general right-handed orthogonal curvilinear coordinate system (q_1, q_2, q_3). Calculate the scale factors for the cylindrical polar coordinate system (ρ, ϕ, z). Calculate and sketch the unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, $\mathbf{e}_z$ relative to Cartesian axes (x, y, z) defined about the same origin as the cylindrical polar coordinate system. [8]

The curl $\boldsymbol{\nabla} \times \mathbf{v} = (v_1, v_2, v_3)$ and the divergence $\boldsymbol{\nabla}.\mathbf{v}$ in a general right-handed orthogonal curvilinear coordinate system (q_1, q_2, q_3) are given by:

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v} &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}\,, \\
\boldsymbol{\nabla}.\mathbf{v} &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 v_1) + \frac{\partial}{\partial q_2}(h_3 h_1 v_2) + \frac{\partial}{\partial q_3}(h_1 h_2 v_3)\right]\,.
\end{aligned}$$

Consider the specific example $\mathbf{u} = u_0(1 - \rho^2)\mathbf{e}_z$ in cylindrical polar coordinates, (where u_0 is a positive constant) defined in the cylindrical region $\rho \leqslant 1$. Using the above formulae, calculate $\boldsymbol{\nabla} \times \mathbf{u}$, $\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u})$ and $\boldsymbol{\nabla} \times (\mathbf{u} \times [\boldsymbol{\nabla} \times \mathbf{u}])$, and hence deduce that

$$([\boldsymbol{\nabla} \times \mathbf{u}].\boldsymbol{\nabla})\,\mathbf{u} = (\mathbf{u}.\boldsymbol{\nabla})\,(\boldsymbol{\nabla} \times \mathbf{u})\,.$$

[6]

2A

Consider the problem for $u(x,t)$ defined on $[0,\pi]$ for $t \geqslant 0$:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \beta(u - u_0)\,; \quad u(x,0) = f(x)\,,\ u(0,t) = u_1\,,\ u(\pi,t) = u_2\,,$$

where $\beta > 0$, u_0, u_1 and u_2 are constants.

By means of the substitution $u(x,t) = u_0 + v(x,t)e^{-\beta t}$, show that $v(x,t)$ satisfies the diffusion equation on $[0,\pi]$ for $t \geqslant 0$:

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}\,,$$

with boundary conditions and initial condition given by

$$v(0,t) = e^{\beta t}(u_1 - u_0)\,, \quad v(\pi,t) = e^{\beta t}(u_2 - u_0)\,, \quad v(x,0) = f(x) - u_0\,.$$

[6]

Now consider the specific situation where $u_0 = u_1 = u_2 \neq 2$, and $f(x) = x(\pi - x)$. Use the method of separation of variables to construct the solution for $v(x,t)$. [10]

Hence show that as $t \to \infty$, $u(x,t)$ may be approximated as

$$u(x,t) \simeq u_0 - \frac{4}{\pi}\,[u_0 - 2]\, e^{-(\beta+1)t} \sin x\,.$$

[4]

3A

Find the general solution $y(x)$ to the homogeneous second order linear differential equation

$$\frac{d^2y}{dx^2} + \frac{4}{x}\frac{dy}{dx} - \frac{4}{x^2}y = 0\,.$$

[4]

Using a Green's function, find an integral representation for the solution of the following inhomogeneous problem:

$$\frac{d^2u}{dx^2} + \frac{4}{x}\frac{du}{dx} - \frac{4}{x^2}u \quad = \quad f(x)\,,\ 0 < x < 1\,,\ u(0) = u(1) = 0\,.$$

[10]

Use this representation to find an explicit solution in the case that

$$f(x) = \begin{cases} x, & 0 < x < 1/2\,, \\ 0, & 1/2 < x < 1\,. \end{cases}$$

[6]

[*Hints: It may be useful to consider solutions of the form* $y = x^n$*, and to evaluate* $u(x)$ *for* $x < 1/2$ *and* $x > 1/2$ *separately.*]

4A

The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ is defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx}\, f(x)\, dx\,.$$

For an appropriately well-behaved function $f(x)$ such that all relevant integrals exist, define the inverse Fourier transform and the autocorrelation $h(x) = f \otimes f$. [4]

Using the definition of $h(x)$, prove Parseval's theorem for Fourier transforms:

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\, dk\,.$$

[6]

Now consider the specific function

$$f(x) = \begin{cases} \cos x & \text{if } |x| \leqslant \pi/2\,, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\tilde{f}(k) = \frac{2\cos\frac{k\pi}{2}}{1-k^2}\,,$$

and hence evaluate the integral

$$\int_{0}^{\infty} \frac{\cos^2 t}{\left(\frac{\pi^2}{4} - t^2\right)^2}\, dt\,.$$

[10]

5B

State the condition(s) for a square $n \times n$ matrix to be *invertible*. [1]

Let $\mathbf{A}$ be a square $n \times n$ complex matrix such that $A_{ij} = 0$ if $i < j$, i.e. it is a lower triangular matrix. Prove by induction or otherwise that the determinant of this matrix is

$$\det\mathbf{A} = A_{11}A_{22}...A_{nn}.$$

[8]

Let $\mathbf{B}$ be an $n \times n$ dimensional invertible complex diagonal matrix with diagonal elements $\{b, b^2, b^3, \ ... \ , b^n\}$ where b is a complex number. Find the condition on b such that the determinant of $\mathbf{B}$ is pure imaginary. [5]

Let $\mathbf{C}$ and $\mathbf{D}$ be two anti-Hermitian matrices, i.e. $\mathbf{C}^\dagger = -\mathbf{C}$, $\mathbf{D}^\dagger = -\mathbf{D}$. Show that $\mathbf{CD}$ is anti-Hermitian if and only if $\mathbf{CD} + \mathbf{DC} = 0$. Find a number α (real or complex) such that $\mathbf{CD} + \alpha\mathbf{DC}$ is anti-Hermitian. [6]

6B

What is the condition on a square matrix $\mathbf{A}$ for it to be *diagonalizable*? [2]

Given the following real 2×2 matrix,

$$\mathbf{A} = \begin{pmatrix} 4 & 5 \\ 1 & 0 \end{pmatrix},$$

find its eigenvalues λ_1, λ_2 and their corresponding eigenvectors $\mathbf{V}_1, \mathbf{V}_2$, and then construct the rotation matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$. [6]

Show that

$$\mathbf{P}^{-1}\mathbf{A}^n\mathbf{P} = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}. \qquad (*)$$

[4]

By assuming that $\mathbf{A}^n = \alpha_n\mathbf{A} + \beta_n\mathbf{I}$ for some scalars α_n and β_n, or otherwise, use $(*)$ to prove that

$$\mathbf{A}^n = \left[\frac{5^n + (-1)^{n+1}}{6}\right]\mathbf{A} + \left[\frac{5^n + 5(-1)^n}{6}\right]\mathbf{I}.$$

[8]

7B

State the Cauchy-Riemann equations of an analytic function

$$f(z) = u(x, y) + iv(x, y)\,,$$

where $z = x+iy$ and x, y are real numbers. Show that $u(x, y)$ and $v(x, y)$ satisfy Laplace's equation. [4]

What is an *entire function*? Prove that $f(z) = \sinh(z)$ is an entire function. By induction, prove that $f(z) = z^n$, where n is a positive integer, is also an entire function. [10]

What are the conditions on the function $f(z)$ to have a pole of order n at z_0? What does it mean for the function $f(z)$ to have an *essential singularity* at z_0? Show that $f(z) = e^{1/z}$ has an essential singularity at $z = 0$. For the function

$$f_N(z) = z^N e^{1/z}\,,$$

where N is a positive integer, calculate the Laurent series about the point at infinity. Deduce that $f_N(z)$ has a pole at $z = \infty$, and identify its order. [6]

8A

Define an *ordinary point*, a *regular singular point* and an *irregular singular point* of a second order linear ordinary differential equation. [3]

Laguerre's equation for $y(x)$ is defined as

$$x\frac{d^2y}{dx^2} + (1 - x)\frac{dy}{dx} + \nu y = 0\,,$$

where ν is a real constant. Show that $x = 0$ is a regular singular point, and $x = \infty$ is an irregular singular point. [4]

Search for solutions of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+\sigma}\,,$$

where σ is a (not necessarily integer) constant to be determined, and $a_0 \neq 0$. Show that the indicial equation has a double root $\sigma^2 = 0$, and hence derive the recursion relation for the coefficients of the power series solution to Laguerre's equation. Briefly comment on the convergence properties of the series. [6]

Assume that $a_0 = 1$, and that $\nu = n$, a positive integer. Show that the recursion relation terminates, thus defining the nth Laguerre polynomial $L_n(x)$. Compute $L_0(x)$, $L_1(x)$, $L_2(x)$ and $L_3(x)$. [7]

9A

State Euler's equation for determining stationary values of functionals $I[y]$ of the form

$$I[y] = \int_{x_s}^{x_e} F(x, y, y')\, dx \,,$$

along paths $y(x)$ between fixed points (x_s, y_s) and (x_e, y_e), and hence show that if $F(y, y')$ does not depend explicitly on x then Euler's equation reduces to

$$F - y' \frac{\partial F}{\partial y'} = A \,,$$

where A is a constant, and a prime denotes differentiation with respect to x. [4]

One particular form of Fermat's principle states that the path taken by a ray of light between two points in a medium is the path which makes the elapsed time stationary. Consider a medium where the speed of light $c(y)$ is a function of y alone. By taking a first integral of the Euler equation or otherwise, show that the rays of light follow paths defined implicitly by the equation

$$\int^{y} \frac{Ac(\hat{y})}{\left(1 - A^2[c(\hat{y})]^2\right)^{1/2}}\, d\hat{y} = \pm(x + B) \,,$$

where A and B are constants to be determined by requiring the path to pass through the start and end points. [8]

Now consider a specific medium filling the upper half-plane $y > 0$ where $c(y) = 1/y$ and the start and end points of interest are $(-1, \cosh[1])$ and $(1, \cosh[1])$ respectively. Calculate the path followed by a ray of light travelling between the start and end points, the minimal value of y along this path, and the time taken to travel between the start and end points. [8]

[*Hint: It may be convenient to use the addition formulae for hyperbolic functions.*]

10A

Consider the self-adjoint problem for $y(x)$:

$$\frac{d^2y}{dx^2} + (\lambda_\epsilon - \epsilon x)y = 0\,, \quad 0 < x < \pi\,,\ y(0) = y(\pi) = 0\,, \qquad (*)$$

where $\epsilon \geqslant 0$, and λ_ϵ is a (real) constant.

Consider the functional

$$F[u] = \int_0^\pi \left[(u')^2 + \epsilon x u^2\right] dx\,, \qquad (\dagger)$$

where $u(x)$ are members of the class of functions such that $u(0) = u(\pi) = 0$ and

$$G[u] = \int_0^\pi u^2\, dx = 1\,,$$

while a prime denotes differentiation with respect to x. Show that stationary values of F correspond to eigenvalues λ_ϵ of $(*)$, and the functions which make F stationary correspond to the associated eigenfunctions of the eigenvalues λ_ϵ. [8]

When $\epsilon = 0$, show that the smallest eigenvalue for the problem $(*)$ is $\lambda_0 = 1$ with associated (normalised) eigenfunction $Y_1(x)$ given by

$$Y_1(x) = \sqrt{\frac{2}{\pi}}\sin x.$$

[4]

Using $Y_1(x)$ as a trial function for $u(x)$ in $(\dagger)$, calculate an upper bound for the smallest eigenvalue λ_ϵ of the full problem $(*)$ for non-zero ϵ. [8]

[*Hints: The Euler equation can be used without proof, and it may be convenient to express* $\sin^2 x$ *using a double angle formula.*]

END OF PAPER