UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday 29 May 2007 9 to 12

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6A****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

For the spherical polar co-ordinates (r, θ, ϕ) defined by

$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta, \qquad r > 0, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi < 2\pi$$

the gradient $\boldsymbol{\nabla}$ and the Laplacian ∇^2 are given by the following:

$$\boldsymbol{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \frac{\mathbf{e}_\theta}{r}\frac{\partial}{\partial \theta} + \frac{\mathbf{e}_\phi}{r\sin\theta}\frac{\partial}{\partial \phi},$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] .$$

Let

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} .$$

(i) Show that

$$\boldsymbol{\nabla} \times \hat{\mathbf{r}} = 0, \qquad \boldsymbol{\nabla} \cdot \hat{\mathbf{r}} = \frac{2}{r} .$$

[5]

(ii) Let

$$\mathbf{f}(\mathbf{r}) = R\hat{\mathbf{r}} + \hat{\mathbf{r}} \times \boldsymbol{\nabla} F + (\hat{\mathbf{r}} \times \boldsymbol{\nabla} G) \times \hat{\mathbf{r}}$$

where R, F, G are smooth functions of r, θ, ϕ. Show that

(a)

$$\mathbf{f} = R\mathbf{e}_r + \frac{1}{r}\left(\frac{\partial G}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial F}{\partial\phi}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial G}{\partial\phi} + \frac{\partial F}{\partial\theta}\right)\mathbf{e}_\phi .$$

[5]

(b) $(\boldsymbol{\nabla} \times \mathbf{f}) \cdot \mathbf{r}$ is independent of R and G.

[5]

(c)

$$(\boldsymbol{\nabla} \times \mathbf{f}) \cdot \mathbf{r} = \frac{1}{r}\Delta_{\theta,\phi} F$$

for some differential operator $\Delta_{\theta,\phi}$ which should be determined.

[5]

[*You may assume the identity* $\boldsymbol{\nabla} \times (\mathbf{f}_1 \times \mathbf{f}_2) = \mathbf{f}_1 \boldsymbol{\nabla} \cdot \mathbf{f}_2 - \mathbf{f}_2 \boldsymbol{\nabla} \cdot \mathbf{f}_1 + (\mathbf{f}_2 \cdot \boldsymbol{\nabla})\mathbf{f}_1 - (\mathbf{f}_1 \cdot \boldsymbol{\nabla})\mathbf{f}_2$.]

2A The heat flow along a thin circular wire of length $2L$ can be approximated by the heat equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \ , \qquad \kappa > 0, \quad -L < x < L$$

with the following boundary conditions

$$u(-L,t) = u(L,t) \ , \qquad \frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t) \ .$$

(i) Use separation of variables to express the solution u in terms of an infinite Fourier series involving appropriate integral transforms of the initial condition $u(x,0) = u_0(x)$.

[10]

(ii) Compute the integral transforms appearing in (**i**) in the particular case that

$$u_0(x) = \sin x - (\frac{\sin L}{L})x \ .$$

[10]

3A

(i) The cosine transform of a function $f(x)$, which has sufficient smoothness and decay as $x \to \infty$, is given by

$$\hat{f}_c(k) = 2\int_0^\infty f(x)\cos kx \; dx, \qquad k > 0 \; .$$

By applying the Fourier transform to an even function, show that the inverse cosine transform is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty \hat{f}_c(k)\cos kx \; dk, \qquad x > 0 \; .$$

[7]

(ii) Let $f(x,t)$ satisfy the partial differential equation

$$i\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial x^2} = 0, \qquad -\infty < x < \infty, \quad t > 0 \; ,$$

and the initial condition

$$f(x,0) = f_0(x), \qquad -\infty < x < \infty \; ,$$

where the function $f_0(x)$ has sufficient smoothness and decay as $|x| \to \infty$. Use the Fourier transform to show that $f(x,t)$ can be written as

$$f(x,t) = \frac{c}{2\pi\sqrt{t}}\int_{-\infty}^\infty f_0(\xi) e^{\frac{i(x-\xi)^2}{4t}} \; d\xi \; ,$$

where

$$c = \int_{-\infty}^\infty e^{-i\ell^2} d\ell \; .$$

[13]

4C

Define the trace of a square matrix and show that $Tr(AB) = Tr(BA)$. Deduce that there are no $n \times n$ matrices A, B such that

$$AB - BA = I \ ,$$

where I is the identity matrix. [6]

Let A, B be real $n \times n$ matrices such that the complex matrix $C = A + iB$ is invertible. By considering $det(A + \lambda B)$ as a function of λ, show that the matrix $A + \lambda B$ is invertible for some real number λ. [6]

Deduce that if two real matrices P, Q are related by a complex similarity transformation $P = RQR^{-1}$, where R is a complex matrix, then they are also related by a real similarity transformation. [8]

[Hint: for the last part rearrange the similarity relation and consider its real and imaginary parts.]

5C

Define the standard inner product in a complex vector space $\mathbb{C}^n$, and prove the Cauchy–Schwarz inequality. [8]

Show that if U is a unitary matrix, then $|U\mathbf{a}| = |\mathbf{a}|$ for all vectors $\mathbf{a}$. Hence, find a constraint for the eigenvalues of U. [4]

Given a one parameter family of unitary matrices

$$U(t) = I + tA + O(t^2),$$

where t is real, show that the eigenvalues of the matrix A are purely imaginary. [8]

6A

(i) Let $y(z)$ satisfy

$$2z(1-z)y'' + (1+z)y' - y = 0 \ .$$

Find the indicial equation associated with the singular point $z = 1$.

[4]

(ii) Let $\psi(x)$ satisfy the Schrödinger equation

$$\psi'' + (2\lambda + 1 - x^2)\psi = 0 \ .$$

Use the transformation $\psi(x) = e^{-\frac{x^2}{2}} y(x)$ to obtain the equation satisfied by $y(x)$, which is called the Hermite equation. [8]

Construct two linearly independent series solutions of the Hermite equation in the neighbourhood of $x = 0$. Give the radius of convergence of the series obtained and construct the first three terms of these series. Find the particular values of λ for which there exist polynomial solutions. Find such solutions up to terms including x^3.

[8]

7A

(i) Let $y_n(x)$ satisfy

$$\frac{d}{dx}(g(x)\frac{dy_n(x)}{dx}) + h(x)y_n(x) + \lambda_n w(x)y_n(x) = 0 \ , \qquad a < x < b \ ,$$

$$\alpha_1 \frac{dy_n}{dx}(a) + \alpha_2 y_n(a) = 0 \ , \qquad \beta_1 \frac{dy_n}{dx}(b) + \beta_2 y_n(b) = 0 \ ,$$

where α_1, α_2 are not both zero and β_1, β_2 are not both zero. Show that if $\lambda_m \neq \lambda_n$ then y_n and y_m satisfy the following orthogonality relation

$$\int_a^b w(x)y_n(x)y_m(x) \ dx = 0 \ ,$$

for $m \neq n$.

[10]

(ii) Let

$$\frac{d}{dx}(x\frac{dy_n(x)}{dx}) + \frac{\lambda_n}{x}y_n(x) = 0 \ , \qquad 1 < x < b \ ,$$

$$y_n(1) = y_n(b) = 0 \ ,$$

where $\lambda_n \neq 0$. Use the change of variables $\xi = \lambda_n \ln x$ to compute λ_n and $y_n(x)$.

[10]

8A Use an appropriate Green's function to solve the following differential equation:

$$\frac{d^2u}{dx^2} - u = e^x \ , \qquad 0 < x < 1 \ ,$$

$$u(0) = u(1) = 0 \ .$$

[20]

[*You may use the identity* $\sinh a \cosh b - \cosh a \sinh b = \sinh(a-b)$.]

9B

Show that functions $y(x)$ which make stationary the functional

$$F[y] = \int_b^a f(x, y, y')\ dx\ ,$$

where $y' = \frac{dy}{dx}$, and $y(a)$, $y(b)$ are fixed, satisfy Euler's equation

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}\ .$$

[10]

The motion of a particle in a plane is constrained by the Lagrangian

$$L = \frac{\dot{r}^2}{(1 - \frac{2m}{r})} + r^2\dot{\theta}^2 + \frac{\mu}{(1 - \frac{2m}{r})}\ ,$$

where m, μ are positive constants and $r(t), \theta(t)$ are generalized co-ordinates with $r > 2m$, $0 \leqslant \theta < 2\pi$. By setting

$$J = r^2\dot{\theta}\ ,$$

show that the Euler-Lagrange equations imply that J is constant.

By computing the Euler-Lagrange equation for r, show that solutions with $r = R$ for constant $R > 2m$ are only possible when

$$J^2 = \frac{m\mu R}{(1 - \frac{2m}{R})^2}\ .$$

[10]

10B

The Sturm-Liouville equation is given by

$$-\frac{d}{dx}(p(x)y') + q(x)y = \lambda w(x)y \ , \qquad a < x < b \ , \qquad (\star)$$

where λ is constant and $p(x) > 0$, $q(x) > 0$ and $w(x) > 0$. Show that solutions of this equation which satisfy the boundary condition

$$p(b)y(b)y'(b) - p(a)y(a)y'(a) = 0$$

correspond to functions $y(x)$ for which the quotient

$$\Lambda[y] = \frac{F[y]}{G[y]}$$

is stationary, where

$$F[y] = \int_a^b (py'^2 + qy^2)dx \ , \qquad G[y] = \int_a^b \omega y^2 dx \ .$$

Show furthermore that the eigenvalues λ of this Sturm-Liouville problem are given by the values of $\Lambda[y]$, where y satisfies $(\star)$.

[10]

Suppose $y(x)$ satisfies the second order differential equation

$$-y'' - \frac{(n-1)}{x}y' + x^2 y = \lambda y \ , \qquad x > 0 \ ,$$

where n is a fixed positive integer and λ is constant. By writing this equation in the form of a Sturm-Liouville equation with $p(x) = w(x) = x^{n-1}$ and $q(x) = x^{n+1}$, use the Rayleigh-Ritz method with a trial function of the form e^{ax^2} for $a < 0$ to find an estimate for the lowest eigenvalue of the Sturm-Liouville problem.

[10]

[*For this question, you may assume the Euler equation without proof.*]

END OF PAPER