UNIVERSITY OF CAMBRIDGE

# NATURAL SCIENCES TRIPOS Part IB & II (General)

Tuesday 30 May 2006 9 to 12

## MATHEMATICS (1)

### Before you begin read these instructions carefully:

*You may submit answers to no more than* **six** *questions. All questions carry the same number of marks.*

*The approximate number of marks allocated to a part of a question is indicated in the right hand margin.*

*Write on* **one** *side of the paper only and begin each answer on a separate sheet.*

### At the end of the examination:

*Each question has a number and a letter (for example,* ***6A****).*

*Answers must be tied up in* **separate** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

***Do not join the bundles together.***

*For each bundle, a blue cover sheet must be completed and attached to the bundle.*

*A* **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

***Every cover sheet must bear your examination number and desk number.***

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1B**

Let $x, y, z$ be Cartesian co-ordinates, and let $\Phi$ be a scalar field.

Define the gradient $\nabla\Phi$ in Cartesian co-ordinates.

Let $q_1, q_2, q_3$ be orthogonal curvilinear co-ordinates. Show that the gradient of $\Phi$ in orthogonal curvilinear co-ordinates is

$$\nabla\Phi = \frac{\partial\Phi}{\partial q_1}\frac{\mathbf{e}_1}{h_1} + \frac{\partial\Phi}{\partial q_2}\frac{\mathbf{e}_2}{h_2} + \frac{\partial\Phi}{\partial q_3}\frac{\mathbf{e}_3}{h_3}$$

and define the quantities $\mathbf{e}_i$ and $h_i$ which appear in this expression. [8]

Oblate spheroidal co-ordinates $(R, \theta, \phi)$ are defined by

$$\begin{aligned} x &= \cosh R \cos\theta \cos\phi \\ y &= \cosh R \cos\theta \sin\phi \\ z &= \sinh R \sin\theta \ . \end{aligned}$$

Show that the co-ordinate surfaces associated with $R, \theta, \phi$ intersect at right angles. [8]

Show that for these co-ordinates,

$$h_R = h_\theta = \sqrt{\sinh^2 R + \sin^2\theta}, \qquad h_\phi = \cosh R \cos\theta \ .$$

[4]

**2A**

Vibrations of a violin string are governed by the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \ ,$$

where $u(x,t)$ is the deflection of the string and $c$ is a constant. The string is fixed at the ends $x = 0$ and $x = L$. The initial deflection is $f(x)$ and the initial velocity is $g(x)$. Use the method of separation of variables to find the deflection at later times in terms of an infinite series

$$u(x,t) = \sum_{n=1}^{\infty} \left[ a_n F_n(t) + b_n G_n(t) \right] H_n(x) \ ,$$

where $F_n(t)$, $G_n(t)$, $H_n(x)$, $a_n$ and $b_n$ should all be specified. [12]

An initial displacement of the string is given by

$$f(x) = \begin{cases} \dfrac{2kx}{L} & \text{if } 0 < x < \frac{L}{2} \\[2ex] \dfrac{2k(L-x)}{L} & \text{if } \frac{L}{2} < x < L \end{cases}$$

Sketch this function. [2]

Find the solution corresponding to the string being released from rest with the above initial displacement. [6]

**3B**

The Fourier transform of a function $f$ is given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \ .$$

Give the expression for the inverse Fourier transform.

Let $\alpha > 0$ and define for non-negative integer $n$:

$$f_n(x) = \begin{cases} x^n e^{-\alpha x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

Show that

$$\tilde{f}_0(k) = \frac{1}{\alpha + ik}$$

and

$$\tilde{f}_n(k) = \frac{n}{\alpha + ik} \tilde{f}_{n-1}(k) \ , \qquad n = 1, 2, \ldots$$

Hence compute $\tilde{f}_n(k)$. [8]

Prove the following identity (Parseval's theorem) for functions $f$ and $g$:

$$\int_{-\infty}^{\infty} [f(x)]^* g(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k) dk \ ,$$

where $^*$ denotes complex conjugation. [8]

Hence show that

$$\int_{-\infty}^{\infty} \frac{1}{(\alpha^2 + k^2)^{n+1}} dk = \frac{2\pi (2n)!}{(n!)^2 (2\alpha)^{2n+1}} \ .$$

[4]

*[In this question all functions may be assumed to be sufficiently integrable so that their Fourier transforms and inverse Fourier transforms are well-defined.]*

**4B**

**(i)** Let $\mathbf{a} \in \mathbb{R}^n$ be a fixed $n$-component real vector, $\mathbf{a} \neq 0$. Let $A^+$ and $A^-$ be real $n \times n$ matrices with components

$$(A^{\pm})_{ij} = \delta_{ij} \pm a_i a_j \ .$$

Obtain the eigenvalues of $A^{\pm}$ and describe the corresponding eigenvectors. [4]

Show that $A^+$ is always invertible and obtain necessary and sufficient conditions for $A^-$ to be invertible. [4]

**(ii)** Let $\mathbf{b} \in \mathbb{R}^3$ be a fixed vector, $\mathbf{b} \neq 0$. Let $B^+$ and $B^-$ be real $3 \times 3$ matrices with components

$$(B^{\pm})_{ij} = \delta_{ij} \pm b_i b_j + \epsilon_{ijk} b_k \ .$$

By choosing a suitable basis for $\mathbb{R}^3$, or otherwise, determine the real eigenvalues of $B^{\pm}$ and the corresponding eigenvectors. [6]

Show that $B^+$ is invertible and obtain necessary and sufficient conditions for $B^-$ to be invertible. [6]

**5B**

What does it mean for a $n \times n$ square matrix to be *diagonalizable*? [2]

Suppose that $A$ is a $n \times n$ square matrix such that $A^p = 0$ for some positive integer $p$. Show that $A$ has 0 as an eigenvalue. Show also that $A$ cannot be diagonalizable unless $A = 0$. [6]

Let $B$ and $C$ be the matrices

$$B = \begin{pmatrix} 4+2\alpha & -2 & -2-4\alpha \\ 3\alpha & -3 & 9-6\alpha \\ 2+\alpha & -1 & -1-2\alpha \end{pmatrix}$$

for $\alpha \in \mathbb{R}$, and

$$C = \begin{pmatrix} 0 & 2 & 6 \\ 3 & 3 & 3 \\ 3 & 1 & -3 \end{pmatrix}$$

By considering the characteristic polynomials of $B$ and $C$, determine whether $B$ and $C$ are diagonalizable. [12]

**6A**

Define a *singular point* and a *regular singular point* of a homogeneous second-order linear ordinary differential equation. Identify two singular points of the following equation and determine their nature:

$$2x(1-2x)y'' + (12x^2 - 4x + 1)y' - 2(4x^2 + 1)y = 0 \qquad (*)$$

[3]

**(i)** Find two linearly independent solutions to $(*)$ in the form of power series about $x = 0$. The coefficients of the power series should be given in terms of recurrence relations, which you are not required to solve. [10]

**(ii)** Write down the coefficient of the $x^n$ term in the Taylor series expansions about $x = 0$ of $e^x$ and $e^{2x}$. Demonstrate that these series converge for all $x$. [3]

**(iii)** Use your answers to (i) and (ii) to verify that the general solution to $(*)$ can be written as

$$y = A\sqrt{x}e^x + Be^{2x} \ .$$

[4]

**7C** Show how to express the eigenvalue equation

$$\frac{d^2y}{dx^2} + u(x)\frac{dy}{dx} + v(x)y + \lambda w(x)y = 0\,,$$

(where $u(x), v(x), w(x)$ are real functions and $w(x) > 0$ for $a \leqslant x \leqslant b$) in Sturm–Liouville form, [6]

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda r(x)y = 0\,.$$

Suppose that $y$ satisfies the boundary conditions

$$k_1 y(a) + k_2 y'(a) = 0\,, \qquad l_1 y(b) + l_2 y'(b) = 0\,,$$

where $k_1,\ k_2,\ l_1,\ l_2$ are constants. Show that two eigenfunctions $y_m,\ y_n$, with distinct eigenvalues $\lambda_m \neq \lambda_n$ satisfy the orthogonality condition

$$\int_a^b r(x) y_m y_n dx = \frac{1}{\lambda_m - \lambda_n}\big[p(b)(y_n'(b)y_m(b) - y_m'(b)y_n(b)) - p(a)(y_n'(a)y_m(a) - y_m'(a)y_n(a))\big] = 0\,.$$

[4]

Find the eigenfunctions and the values of the eigenvalue $\lambda$ that satisfy the equation

$$y'' + 2\alpha y' + (\alpha^2 + \lambda)y = 0\,,$$

where $y(0) = y(\pi) = 0$. [6]

Put the equation in Sturm–Liouville form and hence determine the orthogonality condition for the eigenfunctions. [4]

**8A**

By first finding the appropriate Green's function, solve the differential equation

$$y''(x) + k^2 y(x) = f(x) \ ,$$

where $k$ is a non-zero real number for the cases:

$$y(0) = y'(0) = 0 \ , \qquad f(x) = 2k \cos kx \ ,$$

[10]

and

$$y(0) = y\left(\frac{\pi}{6k}\right) = 0 \ , \qquad f(x) = \frac{k^2}{2} \ .$$

[10]

[*You may use the identity* $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.]

**9C**

The function $y(x)$ makes the value of

$$F = \int_a^b \left(p(x)\,[y'(x)]^2 - q(x)\,[y(x)]^2\right)\,dx$$

stationary subject to the condition $G = 1$, where

$$G = \int_a^b r(x)\,[y(x)]^2\,dx\ ,$$

and with the boundary conditions $y(a) = y(b) = 0$ (where $p(x), q(x), r(x)$ are real functions and $r(x) > 0$ for $a \leqslant x \leqslant b$). Use Euler's equation to show that $y(x)$ satisfies the Sturm–Liouville equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda r(x)y = 0\,.$$

Show that the stationary values of $\Lambda = F/G$ are eigenvalues corresponding to the values of $\lambda$. [8]

Consider the equation

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) - xy + \lambda y = 0\,,$$

where $y(0)$ is finite and $y \to 0$ in the limit $x \to \infty$, and $\lambda$ denotes an eigenvalue. Use the Rayleigh–Ritz method to obtain an estimate of the smallest eigenvalue by using a trial function $y^{(1)}_{trial} = e^{-\beta x}$. [5]

Now use instead the trial function $y^{(2)}_{trial} = A - x$ for $0 \le x \le A$, $y = 0$ for $A \le x \le \infty$ to give a new estimate of the smallest eigenvalue. [5]

Which of these two estimates is the better and why? [2]

[*You may wish to use the fact that* $\int_0^\infty x\,e^{-kx}\,dx = 1/k^2$ .]

**10C** State the Euler equation obtained by minimizing $\int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx$, with $y(x_1)$ and $y(x_2)$ fixed at the boundaries. [3]

If $f$ is not an explicit function of $x$, show that

$$y' \frac{\partial f}{\partial y'} - f$$

is constant. [3]

A bead slides down a frictionless wire, starting at rest at $x = 0$, $y = 0$ and reaching a point B at $x = x_B$, $y = y_B$ after a time $t$. By considering the bead's total energy, show that its velocity at any point during its motion is given by $v = \sqrt{2gy}$. [4]

Hence, show that the time $T[y]$ taken to reach $B$ depends on the shape of the wire $y(x)$ according to

$$T[y] = \frac{1}{\sqrt{2g}} \int_0^{x_B} \sqrt{\frac{1 + {y'}^2}{y}}\, dx\,.$$

Consider the variational problem that determines the shape of the wire that minimizes the time (the *brachistochrone*) and show that the quantity $y(1 + {y'}^2)$ is constant for such a shape. Hence, determine the parametric equations of the brachistochrone,

$$x = c(\theta - \tfrac{1}{2}\sin 2\theta)\,, \qquad y = c\,\sin^2\theta\,,$$

where $c$ is a constant and $\theta$ parameterizes the curve. [5]

Show that if $x_B = l$ and $y_B = 0$ the minimum time is equal to $\sqrt{2\pi l/g}$. [5]

## END OF PAPER