NATURAL SCIENCES TRIPOS Part IB
NATURAL SCIENCES TRIPOS Part II (General)

Tuesday 25 May 2004 9.00 to 12.00

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than **six** questions. All questions carry the same number of marks.

The approximate number of marks allocated to a part of a question will be indicated in the right-hand margin.

Write on **one** side of the paper only and begin each answer on a separate sheet.

At the end of the examination:

Each question has a number and a letter (for example, **6A**).

Answers must be tied up in **separate** bundles, marked **A, B or C** according to the letter affixed to each question.

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A **separate** yellow master cover sheet listing all the questions attempted **must** also be completed.

Every cover sheet must bear your examination number and desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

[TURN OVER

1B Let u_1, u_2 and u_3 be a set of orthogonal, right-handed, curvilinear coordinates, and let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be the corresponding unit basis vectors such that $\mathrm{d}\mathbf{r} = \sum_{i=1}^{3} h_i \,\mathrm{d}u_i \,\mathbf{e}_i$, where $\mathbf{r}$ is a position vector. Show that

$$\nabla\phi = \frac{\mathbf{e}_1}{h_1}\frac{\partial\phi}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial\phi}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial\phi}{\partial u_3}. \qquad [4]$$

Evaluate ∇u_1 and show that

$$\nabla \times (\phi\,\mathbf{e}_1) = \frac{\mathbf{e}_2}{h_1 h_3}\frac{\partial}{\partial u_3}(\phi h_1) - \frac{\mathbf{e}_3}{h_1 h_2}\frac{\partial}{\partial u_2}(\phi h_1). \qquad [4]$$

Evaluate $\nabla u_2 \times \nabla u_3$ and show that

$$\nabla \cdot (\phi\,\mathbf{e}_1) = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}(\phi h_2 h_3). \qquad [4]$$

Use these expressions to deduce general expressions for $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$, where $\mathbf{v}$ has curvilinear components (v_1, v_2, v_3). [4]

In cylindrical polar coordinates, (r, θ, z), find $\nabla \times \mathbf{v}$ for $\mathbf{v} = (\alpha\theta r^p, r^3, 0)$. Find values of p and α for which the vector field $\mathbf{v}$ is irrotational, i.e. $\nabla \times \mathbf{v} = 0$. [4]

2B Write down the partial differential equation that describes how the temperature, $\theta(x,t)$, evolves in time in a one-dimensional bar of length L and thermal diffusivity ν, where x is the spatial coordinate and t is time. [2]

What are the boundary conditions if (a) the ends of the bar are kept at zero temperature or (b) the ends of the bar are insulated? [4]

Assume that the initial temperature at $t = 0$ is

$$\theta(x,0) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant L/2, \\ L - x & \text{if } L/2 < x \leqslant L. \end{cases}$$

Find the solution to the problem, $\theta(x,t)$, with boundary conditions (a). Also, find the solution to the problem, $\theta(x,t)$, with boundary conditions (b). [12]

Compare the behaviour of $\theta(x,t)$ as $t \to \infty$ in (a) and (b). [2]

3A (a) Define the Fourier transform of a function $f(x)$ and write down the inverse Fourier transform. [3]

(b) If f is the function

$$f(x) = \exp(-|x|)$$

and g is the function

$$g(x) = \begin{cases} 1 & \text{for } |x| < 2\pi \\ 0 & \text{for } |x| \geqslant 2\pi \end{cases},$$

derive the Fourier transforms of $f(x)$ and $g(x)$. [6]

(c) Define the convolution of two functions, and state the convolution theorem. State Parseval's theorem and hence evaluate the integral

$$\int_{-\infty}^{\infty} \left| \frac{\sin k}{k(1+k^2)} \right|^2 \mathrm{d}k. \qquad [11]$$

4A (a) Define a Hermitian matrix, and show that the eigenvalues of a Hermitian matrix are real. [6]

Let $\{\mathbf{u}_i\}$ for $i = 1, ..., n$ be a basis of an n-dimensional vector space. Denote by G the $n \times n$ matrix with entries given by the scalar products

$$G_{ij} = \langle \mathbf{u}_i, \mathbf{u}_j \rangle.$$

Show that G is Hermitian. [4]

(b) Given a matrix A (not necessarily Hermitian) with two distinct eigenvalues, show that the corresponding eigenvectors are linearly independent. Hence or otherwise show that a matrix with n distinct eigenvalues has n linearly independent eigenvectors. [10]

5A Define what is meant by an orthogonal matrix and a unitary matrix. [4]

Give the properties required to define the scalar product $\langle , \rangle$ in a vector space V over the complex numbers $\mathbb{C}$, and show that the scalar product is *antilinear* in the first argument, i.e. for vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{v}$ in V, and complex constants α and β, show that the scalar product obeys

$$\langle \alpha \mathbf{u}_1 + \beta \mathbf{u}_2, \mathbf{v} \rangle = \alpha^* \langle \mathbf{u}_1, \mathbf{v} \rangle + \beta^* \langle \mathbf{u}_2, \mathbf{v} \rangle \, . \qquad [6]$$

Find the eigenvalues and eigenvectors of the matrix A, where

$$A = \begin{pmatrix} 1 & a & 0 \\ a & b & 0 \\ 0 & 0 & c \end{pmatrix} ,$$

a, b and c are real positive constants, and $a = \sqrt{b}$. [10]

6A For the differential equation

$$y'' + p(x)y' + q(x)y = 0 \, ,$$

state under what conditions a point $x = x_0$ is

(i) an ordinary point;

(ii) a regular singular point. [4]

If $y_1(x)$ and $y_2(x)$ are two solutions of the above equation, define the Wronksian, $W(x)$, of y_1 and y_2. Show that if $W \neq 0$ then y_1 and y_2 are linearly independent. [6]

Find power series solutions of the equation

$$4xy'' + 2(1 - x)y' - y = 0$$

about the point $x = 0$, giving the indicial equation and suitable recurrence relations for the coefficients. Find the radius of convergence of the solutions. [10]

7B (a) Define a self-adjoint operator for a scalar product of two functions $f(x)$ and $g(x)$ defined by

$$\langle f, g \rangle = \int_a^b f^* g w \, \mathrm{d}x,$$

where $w(x) > 0$ for $a < x < b$. [2]

(b) Express the following equation for the function $y(x)$ in Sturm-Liouville form

$$(1 - x^2)y'' - xy' + n^2 y = 0.$$

Find the required boundary conditions for the linear operator acting on $y(x)$ to be self-adjoint over the interval $[-1, 1]$. [7]

(c) Find the eigenvalues, λ, and eigenfunctions, $z(x)$, of the equation

$$z'' + 4z' + (4 + \lambda)z = 0, \quad z(0) = z(1) = 0.$$

What is the orthogonality relation for these eigenfunctions? [11]

8B Calculate the Green's function, $G(x, \zeta)$, for the following ordinary differential equation

$$x^2 y''(x) + 2xy'(x) - 2y(x) = f(x), \qquad (*)$$

where $y(x)$ is bounded at $x = 0$ and as $x \to \infty$, and show that it takes the form

$$G(x, \zeta) = \begin{cases} k_- \, x^a \, \zeta^b & \text{if } 0 \leqslant x < \zeta, \\ k_+ \, \zeta^c \, x^d & \text{if } \zeta \leqslant x < \infty, \end{cases}$$

where a, b, c, d, k_- and k_+ are constants you need to identify. [7]

Find the solution $y(x)$ of the equation $(*)$ in terms of $G(x, \zeta)$. [2]

Show that if $0 \leqslant f(x) \leqslant m$, then for $x > 0$

(a) $$-\frac{m}{2} \leqslant y(x) \leqslant 0\,,$$

and

(b) $$-\frac{m}{3x} \leqslant y'(x) \leqslant \frac{m}{3x}\,.$$

[11]

9C State the Euler-Lagrange equation that is equivalent to the condition that the functional

$$I = \int_a^b f(x, y(x), y'(x))\,\mathrm{d}x$$

should be stationary with respect to small variations in the function $y(x)$ which preserve the values of $y(a)$ and $y(b)$. [2]

For the two special cases where f is independent of (a) x or (b) $y(x)$, derive first integrals of the Euler-Lagrange equation. [4]

A skateboarder standing on the rim of a hemispherical ramp of radius R, wishes to reach another point on the rim in the least possible time. Assuming conservation of energy, and adopting spherical polar coordinates with the pole $\theta = 0$ at the bottom of the ramp, show that the time to move along a section of any path $\phi = \phi(\theta)$ from θ_0 to θ_1 is

$$T = \left(\frac{R}{2g}\right)^{\frac{1}{2}} \int_{\theta_0}^{\theta_1} \left(\frac{1 + \phi'^2 \sin^2\theta}{\cos\theta}\right)^{\frac{1}{2}} \mathrm{d}\theta\,. \qquad [7]$$

Find a first integral for this problem. Deduce that when beginning to descend, the skateboarder should initially head for the bottom of the ramp. [7]

10C Outline, with justification, why the eigenvalues of the equation,

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+q(x)y=\lambda w(x)y\,,$$

can be found by making

$$I=\frac{\int_a^b(py'^2+qy^2)\mathrm{d}x}{\int_a^b wy^2\mathrm{d}x}$$

stationary, where $p(x)>0$ and $w(x)>0$ for $a<x<b$ and you may assume that $y(x)$ satisfies suitable boundary conditions at $x=a$ and $x=b$ (which should be stated). [9]

A spherically symmetric wavefunction, $\psi(r)$, of the hydrogen atom satisfies for $r>0$ the differential equation

$$r^2\psi''+2r\psi'+2r\psi+2Er^2\psi=0\,,$$

subject to the boundary conditions $\psi(0)=1$ and $\psi\to 0$ as $r\to\infty$. Use the trial function $\psi=\mathrm{e}^{-\alpha r}$ $(\alpha>0)$ to estimate the value of the lowest eigenvalue, E. Obtain an alternative estimate using the trial function

$$\psi=\begin{cases}1-r/\beta, & 0<r<\beta;\\ 0, & r\geqslant\beta.\end{cases}$$

Which is the better estimate? [11]

[*You may quote the results that*

$$\int_0^\infty s^n\mathrm{e}^{-s}\mathrm{d}s=n!\,,\quad \int_0^\beta s(\beta-s)^2\mathrm{d}s=\tfrac{1}{12}\beta^4\,,\quad \int_0^\beta s^2(\beta-s)^2\mathrm{d}s=\tfrac{1}{30}\beta^5\,.\quad]$$

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