UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IB & II (General)

Monday 28 May 2001 1.30 to 4.30

MATHEMATICS (1)

Before you begin read these instructions carefully:

You may submit answers to no more than **seven** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question will be indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

At the end of the examination:

Each question has a number and a letter (for example, ***6C****).*

Answers must be tied up in ***separate*** *bundles, marked* ***A, B or C*** *according to the letter affixed to each question.*

Do not join the bundles together.

For each bundle, a blue cover sheet must be completed and attached to the bundle.

A ***separate*** *yellow master cover sheet listing all the questions attempted* ***must*** *also be completed.*

Every cover sheet must bear your examination number and desk number.

1A Using Cartesian coordinates show that

$$(\boldsymbol{u}.\nabla)\boldsymbol{u} = \frac{1}{2}\nabla u^2 - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}),$$

where $u = |\boldsymbol{u}|$. Briefly explain why this is true irrespective of the coordinate system used to describe the vectors. [7]

Spherical polar coordinates r, θ, ϕ are related to Cartesian x, y, z by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi \quad \text{and} \quad z = r\cos\theta.$$

Prove that spherical polar coordinates are orthogonal and find the metric coefficients h_r, h_θ and h_ϕ such that

$$|\boldsymbol{ds}|^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_\phi^2 d\phi^2,$$

where the vector $\boldsymbol{ds}$ connects the point (r, θ, ϕ) to $(r + dr, \theta + d\theta, \phi + d\phi)$. [6]

A vector field $\boldsymbol{u} = u(r)\hat{\boldsymbol{\phi}}$, where $\hat{\boldsymbol{\phi}}$ is a unit vector in the direction of increasing ϕ. Show that the radial component of $(\boldsymbol{u}.\nabla)\boldsymbol{u}$ is

$$-\frac{u^2}{r}$$

and find the other two components. [7]

[*You may use the following formulae*

$$\nabla\psi = \frac{1}{h_r}\frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{h_\theta}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{h_\phi}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

and

$$\nabla \times \boldsymbol{A} = \frac{1}{h_r h_\theta h_\phi}\begin{vmatrix} h_r\hat{\boldsymbol{r}} & h_\theta\hat{\boldsymbol{\theta}} & h_\phi\hat{\boldsymbol{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ h_r A_r & h_\theta A_\theta & h_\phi A_\phi \end{vmatrix}.]$$

2A The Fourier transform of $f(x)$ is defined as

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}kx}\, dx.$$

Write down the inverse Fourier transform of $\tilde{g}(k)$. [2]

Evaluate the Fourier transform of $f(x) = \mathrm{e}^{-c|x|}$ $(c > 0)$ and hence or otherwise show that the Fourier transform of $f(x) = \frac{1}{x^2+c^2}$ is

$$\tilde{f}(k) = \frac{\sqrt{2\pi}}{2c} \mathrm{e}^{-c|k|}.$$

[6]

Define the convolution $f * g$ of two functions $f(x)$ and $g(x)$ and deduce the convolution theorem for its Fourier transform. [5]

Hence, by taking the Fourier transform, find the function $g(x)$ in the integral equation

$$\int_{-\infty}^{\infty} \frac{g(y)}{(y-x)^2 + a^2}\, dy = \frac{1}{x^2 + b^2},$$

where $b > a > 0$. [7]

3A Define the Laplace transform $F(p)$ of a function $f(t)$, $t \geq 0$. [2]

Derive expressions for the Laplace transforms of df/dt and $tf(t)$. [5]

Evaluate $F(p)$ when $f(t) = \sin kt$. [3]

Hence, by finding dG/dp or otherwise, solve

$$t\frac{d^2g}{dt^2} + 2\frac{dg}{dt} + \alpha^2 tg = 0, \qquad g(0) = 1, \quad g'(0) = 0.$$

[10]

4B Suppose $\mathbf{A}$ is an $n \times n$ matrix, such that $\mathbf{A}^2 = \mathbf{A}$.

(a) By considering the eigenvalues of $\mathbf{A}$, prove **either**: (i) that $\det \mathbf{A} = 1$ and $\mathrm{Tr}\mathbf{A} = n$; **or** (ii) that $\det \mathbf{A} = 0$ and $\mathrm{Tr}\mathbf{A} = m < n$, where m is an integer. [5]

If $\mathbf{A}\boldsymbol{x} = \boldsymbol{y}$, what is $\mathbf{A}\boldsymbol{y}$? What is the dimension of the space of nonvanishing vectors $\boldsymbol{y}$ for the two cases (i) and (ii) mentioned above? [5]

(b) Construct a 4×4 matrix $\mathbf{P}$ such that its action on an arbitrary vector $\boldsymbol{x}$ is

$$P_{ij} x_j = x_i \; - \; \delta_{i4} \sum_{k=1}^{4} x_k \, .$$

What is $\mathbf{P}^2$, $\det \mathbf{P}$, $\mathrm{Tr}\mathbf{P}$? [5]

Find a set of linearly independent vectors that span the space of vectors $\boldsymbol{y} = \mathbf{P}\boldsymbol{x}$. Construct from them an orthonormal basis for this subspace. [5]

5B Given an $n \times n$ matrix $\mathbf{M}$ and the identity $\mathbf{I}$, show that the matrices $\mathbf{I} + \mathbf{M}$ and $(\mathbf{I} - \mathbf{M})^{-1}$ commute. [4]

For a real antisymmetric matrix $\mathbf{A}$, the matrix $\mathbf{N}$ is defined by:

$$\mathbf{N} \; = \; (\mathbf{I} + \mathbf{A}) \, (\mathbf{I} - \mathbf{A})^{-1} \, .$$

Show that $\mathbf{N}$ is orthogonal. [4]

Show that the eigenvectors of $\mathbf{A}$ are also eigenvectors of $\mathbf{N}$. [3]

Show that the three eigenvalues of a real orthogonal 3×3 matrix are (i) $e^{+i\alpha}$, (ii) $e^{-i\alpha}$ and (iii) $+1$ or -1, where α is real. [4]

Hence show that, when $\mathbf{A}$ and $\mathbf{N}$ are 3×3 matrices, $\det \mathbf{N} = 1$ and that there exists a direction $\boldsymbol{x}$ in which $\mathbf{A}\boldsymbol{x} = 0$. [5]

6C Define an *ordinary point* and a *regular singular point* of the ordinary differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0\,.$$

What are the implications for the existence of a series solution at such points? [6]

Find series solutions about $x = 0$ of the equation

$$y'' + \frac{(1-x)}{2x}y' - \frac{1}{4x}y = 0\,.$$

In particular, determine the indicial equation, the recurrence relations and the radius of convergence of your solutions. [12]

Express one of these solutions in closed form. [2]

7C A differential operator $\mathcal{L}$ is self-adjoint on the interval $a \le x \le b$ if for all pairs of functions $y_1,\ y_2$ satisfying appropriate boundary conditions we have

$$\int_a^b [y_1\mathcal{L}y_2 - y_2\mathcal{L}y_1]\,dx = 0\,.$$

By integrating by parts, show that operators in Sturm-Liouville form

$$\mathcal{L}y \equiv -\left[p(x)\,y'\right]' + q(x)y$$

are self-adjoint, given suitable boundary conditions. Specify several examples of these boundary conditions. [6]

Consider the eigenvalue problem

$$(1-x^2)y'' - x\,y' + n^2y = 0$$

on the interval $-1 \le x \le 1$ with given eigenvalues $\lambda_n = n^2$, n an integer.

By considering the substitution $x = \cos\theta$, find the corresponding eigenfunctions

$$y_n(x) = N_n \sin\left[n\,\cos^{-1}x\right]$$

which satisfy $y_n(1) = 0$ and $y_n(-1) = 0$. [6]

Hence (or otherwise) find the weight function $w(x)$ and the normalization constants N_n such that

$$\int_{-1}^{1} w(x)\,y_n(x)\,y_m(x)\,dx \;=\; \delta_{nm}\,.$$

[8]

8C Find the Green's function $G(x, \xi)$ satisfying

$$\frac{d^2G}{dx^2} - \kappa^2 G = \delta(x - \xi), \qquad 0 \leq x \leq 1\,, \qquad 0 \leq \xi \leq 1\,, \qquad (\kappa \text{ real})$$

subject to the boundary conditions $\frac{dG}{dx}(0, \xi) = \frac{dG}{dx}(1, \xi) = 0$. [10]

Hence show that the solution of the equation

$$\frac{d^2y}{dx^2} - \kappa^2 y = f(x),$$

for the boundary conditions $y'(0) = y'(1) = 0$, is given by

$$y(x) = \frac{-1}{\kappa \sinh \kappa}\Big[\cosh \kappa x \int_x^1 f(\xi) \cosh \kappa(1 - \xi) d\xi + \cosh \kappa(1 - x) \int_0^x f(\xi) \cosh \kappa\xi \, d\xi\Big]\,.$$

[4]

Find the explicit solution for $y(x)$ given $f(x) = x$ and $\kappa = 1$. [6]

9C From the Euler-Lagrange equation, $\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$, which extremizes the functional $I = \int_a^b F(x, y, y')dx$, recast the Sturm-Liouville eigenvalue problem

$$-\left[p(x)\, y'\right]' + q(x)y - \lambda w(x)y = 0$$

in variational form, given appropriate boundary conditions for y at $x = a, b$. Hence, explain the Rayleigh-Ritz method for estimating the lowest eigenvalue. [8]

Consider the differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\psi}{dr}\right) + \lambda\psi = 0$$

subject to the boundary conditions $y(1) = 0$ and $y'(0) = 0$.

Find a suitable quadratic trial function satisfying these boundary conditions and apply the Rayleigh-Ritz method to estimate the lowest eigenvalue λ_0. [9]

The actual value is $\lambda_0 = \pi^2$. Why should your estimate be higher? [3]

10C The Euler-Lagrange equation

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0$$

is satisfied by the function $y(x)$ which makes the integral $I = \int_a^b F(x, y, y')dx$ stationary, subject to appropriate boundary conditions.

Show that if $F(x, y, y')$ does not depend explicitly on x, then y also satisfies the first integral

$$F - y'\frac{\partial F}{\partial y'} = k \qquad \text{(const.)}$$

[6]

An optical medium in the planar strip $0 < y \leq 1$ has a variable refractive index $\mu(x, y) = y^{-1}$, with a constant $\mu = 1$ above it ($y > 1$).

Apply Fermat's principle to find the set of paths followed by light rays within the optical medium ($0 < y \leq 1$). [8]

For a light ray entering the medium at $y = 1$ and subtending an angle θ from the y-axis, calculate the distance in the x-direction that the ray will travel; what is the maximum distance possible? [6]

11C We wish to solve numerically the first-order differential equation $y' = f(y, t)$ using one of the following schemes:

$$\text{Euler:} \qquad y_{n+1} = y_n + \Delta t\, f(y_n, t_n)\,,$$

$$\text{Backward Euler:} \quad y_{n+1} = y_n + \Delta t\, f(y_{n+1}, t_{n+1})\,,$$

By Taylor expanding y and f, show that both schemes have a local truncation error $\mathcal{O}(\Delta t^2)$. [6]

By considering the equation $y' = -\Lambda y$ ($\mathrm{Re}\,\Lambda > 0$) (or otherwise), examine the stability of both schemes. [6]

A predictor-corrector method employs both the Euler and Backward Euler schemes as follows:

$$y_{n+1} = \frac{1}{2}\left[y_{n+1}^{(1)} + y_{n+1}^{(2)}\right]\,,$$

where

$$y_{n+1}^{(1)} = y_n + \Delta t\, f(y_n, t_n)\,,$$

$$y_{n+1}^{(2)} = y_n + \Delta t\, f(y_{n+1}^{(1)}, t_{n+1})\,,$$

Find the local truncation error for this method. [4]

Examine the stability of this method. [4]