UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday, 15 June, 2011 9:00 am to 12:00 pm

MATHEMATICS (2)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

*You may submit answers to **all** of section A, and to no more than **five** questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

*Tie up **all of your section A answer** in a single bundle, with a completed blue cover sheet.*

*Each section B question has a number and a letter (for example, **11S**). Answers to each question must be tied up in **separate** bundles and marked (for example, **11S, 12X** etc) according to the number and letter affixed to each question. **Do not join the bundles together**. For each bundle, a blue cover sheet **must** be completed and attached to the bundle, with the correct number and letter written in the section box.*

*A **separate** green master cover sheet listing all the questions attempted **must** also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Determine the constant c so that the vectors $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{v} = 4\mathbf{i} + c\mathbf{j}$ are perpendicular. [2]

2

Evaluate the indefinite integral

$$\int \frac{2x+3}{x^2+3x-2}\,\mathrm{d}x.$$

[2]

3

Find the stationary points of the function

$$y = \frac{2x^2-5x-25}{x^2+x-2}.$$

[2]

Locate the points of discontinuity. [1]

4

Verify that if

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix},$$

then $A^2 - 4A - 5I = 0$, where I is the identity matrix. [2]

5

Find the real and imaginary parts of the complex number

$$\frac{5-2i}{3i+4}.$$

[2]

6

Give the first three non-zero terms of the Taylor expansion of the function $f(x) = \sqrt{x}$ about the point $x = 1$. [2]

7

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xe^y}{x^2+1} \quad \text{with} \quad y(0) = 0.$$

[2]

8

Sketch the curve represented by the equation $|z - 1| = 2$ for complex z. [2]

9

Determine the equation of the line that is tangent to the curve $y = x^3 + 1$ at $x = 2$. [2]

10

Calculate the divergence of the vector field $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$. [1]

SECTION B

11R

Consider vector fields $\mathbf{v}(\mathbf{x})$ and $\mathbf{w}(\mathbf{x})$, where

$$\mathbf{v}(\mathbf{x}) = \nabla\phi(\mathbf{x})$$

and

$$\mathbf{w}(\mathbf{x}) = \big((1-x^2-z^2)\arcsin y,\ (1-x^2-z^2)\mathrm{arctanh}(x^2),\ x\big)\,.$$

Here, $\phi(\mathbf{x}) = e^{|\mathbf{x}|^2}$ and $\mathbf{x} = (x,y,z)$. If Γ_1 and Γ_2 are curves described by $\mathbf{x} = (\cos\theta, e^{-\cos^2\theta}, \sin\theta)$, find the values of

(a) (i) $\oint_{\Gamma_1}(3\mathbf{v}+\mathbf{w})\cdot d\mathbf{x}$, where $0 \leqslant \theta \leqslant 2\pi$.

(ii) $\int_{\Gamma_2}(\nabla\wedge\mathbf{v})\cdot d\mathbf{x}$, where $0 \leqslant \theta \leqslant \pi$

(iii) The line integral $\int_{\Gamma_2}\nabla\cdot[\nabla\wedge(|\mathbf{v}|^2\mathbf{w})]ds$, where $0 \leqslant \theta \leqslant \pi$ [12]

(b) (i) $\int_S(\nabla\cdot\mathbf{v})\ \mathbf{dS}$, where S is the surface of the hemisphere of unit radius centred on the origin with $z \leqslant 0$.

(ii) $\int_{S'}(\nabla\cdot\mathbf{v})\ \mathbf{dS}$, where S' is the surface of the hemisphere of unit radius centred on the origin with $z > 0$. [8]

12S

(a) Write down the Cartesian coordinates (x, y, z) of a point with spherical polar coordinates (r, θ, ϕ). [2]

(b) Show that the distance δs between two neighbouring points (θ, ϕ) and $(\theta + \delta\theta, \phi + \delta\phi)$ on a sphere of radius r is given approximately by

$$\delta s^2 = r^2(\delta\theta^2 + \sin^2\theta\, \delta\phi^2),$$

for sufficiently small $\delta\theta$ and $\delta\phi$. [6]

(c) A path (on the sphere) of minimal distance between a pair of points, A and B, is known to be an arc of a great circle. A great circle is defined to be the intersection of the sphere with a plane through its centre. Show that a great circle obeys an equation which we can choose to be

$$l \sin\theta \cos\phi + m \sin\theta \sin\phi + n \cos\theta = 0,$$

where $l^2 + m^2 + n^2 = 1$. [5]

(d) Take point A to have $\theta = \frac{\pi}{2}, \phi = 0$, and point B to have $\theta = \theta_0, \phi = \phi_0$, where $0 < \theta_0 < \frac{\pi}{2}$ and $0 < \phi_0 < \pi$. Show that the shortest path (on the sphere) between A and B obeys

$$\tan\theta \sin\phi = \tan\theta_0 \sin\phi_0 \,.$$

[7]

13Z

(a) Show that

$$\frac{1}{\cosh x} < \frac{1}{\sinh x} < \coth x \text{ for } x > 0.$$

[6]

(b) Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector $\mathbf{u}$ whose y component is positive and which makes an angle of $\pi/3$ with the positive x axis. [8]

(c) Sketch $f(x, y)$ defined in (b) on the (x, y) plane by showing contours. Clearly display the locations and nature of any stationary points. [6]

14Y

(a) Consider the 3×3 matrix A where

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} .$$

(i) Compute det A. [2]

(ii) Show that the three eigenvalues are 5, -3, -3. [4]

(iii) Determine the three eigenvectors. [4]

(iv) Write down which of the eigenvectors are orthogonal. [2]

(b) C and B are real non-zero 3×3 matrices which satisfy the equation

$$(CB)^T + B^{-1}C = 0$$

(i) Show that if B is orthogonal then C is antisymmetric. [3]

(ii) Without assuming that B is orthogonal, prove that C is singular if $\det(B^T) \neq 0$. [5]

15T

The equations of motion of a projectile subject to a resistive force (such as atmospheric drag) proportional to the velocity are

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\gamma \frac{dx}{dt}, \\ \frac{d^2z}{dt^2} &= -\gamma \frac{dz}{dt} - g, \end{aligned}$$

where γ and g are positive constants.

(a) Find the solutions $x(t)$ and $z(t)$ of these equations for a situation where the initial velocity of the projectile

$$(V_{0x}, V_{0z}) = \left(\frac{dx}{dt}, \frac{dz}{dt} \right)_{t=0}$$

and its initial position is $(0, 0)$. [12]

(b) Derive the equation of the projectile's trajectory $z = z(x)$. [8]

16T

The function

$$E(k,\phi) \equiv \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta,$$

defined for $0 \leqslant k \leqslant 1$, is called an elliptic integral of the second kind.

(a) Write

$$\int_{-1/2}^{1/2} \sqrt{\frac{9-4x^2}{36(1-x^2)}}\, dx$$

as an elliptic integral of the second kind. [6]

(b) An ellipse of semi-axes a and b, such that $a > b$, and eccentricity

$$e \equiv \sqrt{\frac{a^2-b^2}{a^2}}$$

has parametric equations

$$\begin{aligned} x &= a \sin\theta, \\ y &= b \cos\theta, \end{aligned}$$

for $0 < \theta < 2\pi$. Show that the length of the perimeter of the ellipse, $L = \int \sqrt{dx^2+dy^2}$, can be written as $L = a\int_0^{2\pi} d\theta \sqrt{1-e^2\sin^2\theta} = 4aE(e,\pi/2)$. [8]

(c) Evaluate the length of the perimeter of a low eccentricity ellipse ($e \ll 1$) up to order e^2 in terms of its semi-axes a and b. [6]

17X

(a) To play chess you need a chess board and some chess pieces. The chess board (on which you put the pieces) has 64 squares, of which 32 are black and 32 are white. There are 32 chess pieces in total, of which 16 are black and 16 are white. Suppose that the chess pieces are placed at random onto the the squares of the chess board. (Note that each square can accommodate *at most* one piece.) Calculate the probability that every piece lands on a square that matches its own colour. [Note: You should leave the answer as a rational function of permutations ${}^{n}P_{r}$, or as a rational function of factorials, or as some mixture of those.] [8]

(b) A History Tripos examination paper contains only two questions, and each carries m marks. "Question 1" has w words in it, and "Question 2" has $3w$ words. Candidates may only answer one question, and may choose freely. It is well known to The Examiners that the probability that a **Lazy** student will choose a given question is inversely proportional to the number of words in it, whereas a **Hard Working** student will show no preference. Given further that

- **Lazy** students get $\frac{1}{3}m$ marks for whichever question they attempt,
- **Hard Working** students get m marks for whichever question they attempt, and
- the expected number of marks for those students choosing "Question 1" is $\frac{3}{7}m$,

Calculate the probability l that a lazy student chooses Question 1. Then calculate the probability p that a randomly chosen history student is **Lazy**. [12]

18Y

(a) Using the polar coordinates r, θ, evaluate

$$\int_0^{\pi/2} \int_0^R r^2 \cos\theta \sin\theta dr\, d\theta$$

[4]

(b) Find the area of the finite region of the plane bounded by the line $y = 2x + 3$ and the curve $y = 6x - x^2$. [8]

(c) Find the volume of the finite 3D region bounded by the paraboloid

$$z = 4 - x^2 - y^2$$

and the plane $z = 0$. [8]

19X*

(a) Verify that $\Phi(u,v) = \frac{2\alpha}{\pi}\arctan(\frac{v}{u})$ satisfies

$$\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2} = 0 \qquad (*)$$

in the region $u > 0$ of the (u,v)-plane, and sketch the contours of constant Φ in this region. [8]

(b) If $u(x,y)$ and $v(x,y)$ are functions of x and y for which $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, and $\Phi(u,v)$ satisfies $(*)$, it can be shown that the function $\Psi(x,y)$ defined by $\Psi(x,y) = \Phi(u(x,y), v(x,y))$ satisfies

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = 0 \qquad (**)$$

in the part of the (x,y) plane for which $u(x,y) > 0$.

Show that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ for the particular choice

$$u(x,y) = \frac{2(1-x)}{(1-x)^2 + y^2} - 1$$

and

$$v(x,y) = \frac{2y}{(1-x)^2 + y^2}.$$

[6]

(c) Using the choice of u and v given in (b), sketch $u(x,y) = 0$ on the (x,y) plane, indicating on your sketch which parts come from regions where $v > 0$ and which come from regions where $v < 0$. Write down the two-dimensional electric potential $\Psi(x,y)$ satisfying $(**)$ inside a circular capacitor having one semi-circular "terminal" ($x^2 + y^2 = 1$, $y > 0$) at $\Psi = +V$ and the other semi-circular "terminal" ($x^2 + y^2 = 1$, $y < 0$) at $\Psi = -V$. [6]

20R*

Consider the functions $L_n(x)$ defined by

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right),$$

for non-negative integers n.

(a) Show that L_n is a polynomial of degree n and find the coefficients of x^{10} and x^9 of $L_{10}(x)$ in terms of factorials. [6]

(b) Let $v = x^n\, e^{-x}$. Verify that

$$x\frac{dv}{dx} = (n-x)v,$$

and differentiate this identity $n+1$ times to show that L_n satisfies the differential equation

$$x\frac{d^2L_n}{dx^2} + (1-x)\frac{dL_n}{dx} + nL_n = 0.$$

[8]

(c) The previous equation can be rewritten as

$$e^x\frac{d}{dx}\left(xe^{-x}\frac{dL_n}{dx}\right) + nL_n = 0.$$

Hence, for any non-negative integers n, m,

$$L_m\, e^x\frac{d}{dx}\left(xe^{-x}\frac{dL_n}{dx}\right) + nL_mL_n = 0,$$
$$L_n\, e^x\frac{d}{dx}\left(xe^{-x}\frac{dL_m}{dx}\right) + mL_mL_n = 0.$$

Use these results to derive the orthogonality relation

$$\int_0^\infty e^{-x}L_m(x)L_n(x)dx = 0$$

for $m \neq n$. [6]

END OF PAPER