UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday 15 June 2005 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

For ***each question*** *you have attempted, attach a* ***blue*** *cover sheet to your answer and write the question number and letter (for example,* ***3B****) in the 'section' box on the cover sheet.*

List all the questions you attempted on the ***yellow*** *master cover sheet.*

Every cover sheet must bear your candidate number and your desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

(a) Determine whether there is a function ϕ such that $\mathbf{F} = \nabla\phi$, in the two cases:

(i)
$$\mathbf{F} = (xz - y)\mathbf{i} + (-x + y + z^3)\mathbf{j} + (3xz^2 - xy)\mathbf{k},$$
[3]

(ii)
$$\mathbf{F} = \exp(z - x)\Big[(y - yx)\mathbf{i} + x\mathbf{j} + xy\mathbf{k}\Big] + 2x\mathbf{i}.$$
[3]

In each case, calculate the function ϕ, if it exists. [4]

(b) Find the acute angle between the surfaces $x^2y^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point $(1, -2, 1)$. [10]

2A Let $\mathbf{r}$ be a vector of length $r = |\mathbf{r}|$ from the origin of coordinates to a general point P, so that $\mathbf{n} = \mathbf{r}/r$ is a unit vector in the outward radial direction through P. Let the vector field $\mathbf{F}$ be given by
$$\mathbf{F} = \frac{\mathbf{n}}{r^\gamma}$$
for some number $\gamma \neq 2$. Explicitly calculate the three-dimensional volume integral $\int_V (\nabla \cdot \mathbf{F}) dV$ where V is the solid region defined by
$$0 < \epsilon < r < R.$$
[8]

Explicitly calculate the two-dimensional surface integral $\int_S \mathbf{F} \cdot \mathbf{dA}$ over the boundary S of this region.

[8]

If $\gamma = 2$, what are the values of the volume and surface integrals? [4]

[*You may not use the divergence theorem.*]

3B

(a) For what values of α does the following equation have non-zero solutions in (x, y, z)? [6]

$$\begin{pmatrix} 2 & 2-\alpha & 1 \\ 1 & 2+\alpha & 4 \\ -\alpha & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} .$$

Find expressions for x and y in terms of z for each such value of α. [4]

(b)

(i) Let

$$\mathbf{A} = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and let

$$\mathbf{S} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n,$$

where $\mathbf{A}^0 = \mathbf{I}$. Find an expression for $\mathbf{S}$ in terms of ϵ, trigonometric functions of ϵ and the matrices $\mathbf{I}$ and $\mathbf{A}$. [5]

(ii) Similarly, if

$$\mathbf{B} = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}$$

and

$$\mathbf{T} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{B}^n,$$

with $\mathbf{B}^0 = \mathbf{I}$, find an expression for $\mathbf{T}$ in terms of ϵ, hyperbolic functions of ϵ and the matrices $\mathbf{I}$ and $\mathbf{B}$. [5]

4B*

(a) A "Pringle crisp" can be defined as the surface

$$z = f(x, y) = x^2 - y^2$$

with a boundary defined by the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0.$$

Use the method of Lagrange multipliers to find the largest and smallest values of z on the boundary of the crisp, and the (x, y) positions where these occur. [4]

(b) Use the method of Lagrange multipliers to find the point on the line $y = mx + c$ which is closest to the point (x_0, y_0). [6]

(c) A farmer wishes to construct a grain silo in the form of a hollow vertical cylinder of radius r and height h with a hollow hemispherical cap of radius r on top of the cylinder. The walls of the cylinder cost £x per unit area to construct and the surface of the cap costs £$2x$ per unit area to construct. Given that a total volume V is desired for the silo, what values of r and h should be chosen to minimise the cost? [10]

5C

(a) Find the modulus and argument of the complex number $1 + i$ and show where this is on the Argand diagram. If $z^2 = 1 + i$, find $|z|$ and $\arg(z)$ and illustrate your result on the Argand diagram. [4]

(b) Express $\sin 5\theta$ in terms of $\sin\theta$ and find the values of θ such that

$$16 \sin^5 \theta = \sin 5\theta\,.$$

[8]

(c) Show that the solutions of the equation

$$\sin z = 2i \cos z$$

are given by

$$z = (\pi/2 + n\pi) + \frac{i}{2} \ln 3,$$

where n is an arbitrary integer. [8]

6C

(a) What are the vector and Cartesian equations for the surface of a sphere with radius $r = 4$ and centre at position $\mathbf{c} = (1, 2, 3)$? [3]

(b) The Cartesian equation for a plane is

$$x + 2y + 2z - 20 = 0.$$

What is the vector equation for this plane? [3]

(c) Find the shortest distance between $\mathbf{c}$ and this plane. [4]

(d) Find the centre and radius of the circle of intersection of the plane in (b) with the sphere in (a). [10]

7D

(a) The internal energy of a system, $U(S,V)$, has the differential

$$dU = TdS - p\, dV\,.$$

Find a function F such that

$$dF = -SdT - p\, dV$$

[3]

and prove that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V\,.$$

[3]

Given that dS can be written as

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV\,,$$

consider changes in S with respect to T at constant p, and hence show that

$$\left(\frac{\partial S}{\partial T}\right)_p - \left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial p}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_p\,.$$ [4]

(b) By considering the differentials of $S(T,p)$ and $S(T,V)$, respectively, when $dS = 0$, show that

$$\left(\frac{\partial S}{\partial T}\right)_p = -\left(\frac{\partial S}{\partial p}\right)_T \left(\frac{\partial p}{\partial T}\right)_S \quad \text{and that} \quad \left(\frac{\partial S}{\partial T}\right)_V = -\left(\frac{\partial S}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_S\,.$$

[6]

Hence prove that

$$\left(\frac{\partial S}{\partial T}\right)_p \left(\frac{\partial p}{\partial V}\right)_T = \left(\frac{\partial p}{\partial V}\right)_S \left(\frac{\partial S}{\partial T}\right)_V\,.$$ [4]

8D Determine which of the following differentials is exact. In each case, if the differential is exact, find the function $f(x, y)$ such that

$$df = Pdx + Qdy.$$

If the differential is not exact, find an integrating factor.

(a)
$$(12x + 5y - 9)dx + (5x + 3y - 4)dy$$

[6]

(b)
$$(x + y)dx + 2xdy$$

[7]

(c)
$$\frac{(x^2 - y^2)dx + 2xydy}{(x^2 + y^2)^2}$$

[7]

9E

(a) A real square matrix M is said to be *normal* if $MM^T = M^TM$. Show that a real square matrix is normal if it is symmetric, antisymmetric or orthogonal. [4]

(b) Find the eigenvalues of the matrix

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

[5]

Find the corresponding normalized eigenvectors, simplifying your results where possible. [8]

(c) Verify that the eigenvectors are orthogonal. [3]

10E*

(a) Let $f(x)$ be a function that is continuous, positive and decreasing for $x > 1$. Explain why

$$\sum_{n=r+1}^{s} f(n) < \int_r^s f(x)\,dx < \sum_{n=r}^{s-1} f(n),$$

where r and s are any positive integers with $r < s$. [6]

(b) Hence show that

$$\frac{1}{p}\left(1 - N^{-p}\right) + N^{-1-p} < \sum_{n=1}^{N} n^{-1-p} < \frac{1}{p}\left(1 - N^{-p}\right) + 1,$$

where p is any positive number and an integer $N > 1$. [6]

(c) Deduce that the series

$$\sum_{n=1}^{\infty} n^{-1-p}$$ [5]

converges for any $p > 0$, and show further that

$$\lim_{p \to 0}\left(p \sum_{n=1}^{\infty} n^{-1-p}\right) = 1,$$

where p tends to zero through positive values. [3]

11F

(a) By expressing the integrand in partial fractions, evaluate

$$\int_1^2 \frac{3x^2+5x+1}{x(x+1)(x+2)}\,dx.$$

[7]

(b) Evaluate the definite integral

$$\int_0^{\pi/4} \frac{1}{1+\cos 2\theta}\,d\theta.$$

[6]

(c) Using your results from (b), or otherwise, evaluate the definite integral

$$\int_0^{\pi/2} \frac{1}{1+\sin\phi}\,d\phi\,.$$

[7]

12F* The diffusion equation is

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}.$$

Show that a solution to this equation is

$$f(x,t) = \frac{1}{\sqrt{t}} e^{-x^2/4t}.$$

[6]

Suppose that the equation is modified to

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t} - \frac{f}{t}.$$

The solution is now of the form

$$f(x,t) = g(t)\frac{1}{\sqrt{t}} e^{-x^2/4t}.$$

Find the first order ordinary differential equation satisfied by $g(t)$. [9]

Now find the general solution of this first order differential equation. [5]

END OF PAPER