UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday 11 June 2003 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than **six** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Each question has a number and a letter (for example **3B***).*

Answers must be tied up in **separate** *bundles, marked* **A, B, C, D, E** *or* **F** *according to the letter affixed to each question.* **Do not join the bundles together.**

For each bundle, a blue cover sheet **must** *be completed and attached to the bundle, with the appropriate letter written in the section box.*

A **separate** *yellow master cover sheet listing all the questions attempted* **must** *also be completed.*

Every cover sheet must bear your examination number and desk number.

1A

(a) Find the equation of the plane that is perpendicular to the vector $-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and passes through the point A (2, −1, 1). Determine the shortest distance of this plane from the origin. [8]

(b) A second plane passes through the points A, B (1, 1, −1) and C (3, −1, 2). Find a unit vector normal to this plane, and a vector along the line of intersection of the two planes. [12]

2A Show that there are no constant or cosine terms in the Fourier series for an odd function over the range $-\pi \leqslant x \leqslant \pi$. [8]

Find the Fourier series for the function $f(x) = \sin ax$, where $-\pi \leqslant x \leqslant \pi$ and a is not an integer. [12]

[You may find the following trigonometric identities useful:

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi,$$
$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi.]$$

3B

(a) Express the real matrix

$$M = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}$$

as the sum of a real symmetric matrix, S, and a real antisymmetric matrix A.

Determine the values of a and b for which both S and A are orthogonal matrices. [12]

(b) Determine the eigenvalues and normalised eigenvectors of the matrix

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}.$$

Verify that the eigenvectors are orthogonal. [8]

4B*

(a) State carefully the divergence theorem and Stokes' theorem. [4]

(b) In Cartesian coordinates and components, the vector field $\mathbf{F}$ is given by

$$\mathbf{F} = (x^2yz,\ xy^2z,\ xyz^2)\,.$$

Evaluate $\int_S \mathbf{F}\cdot d\mathbf{S}$, where S is the surface of the cube

$$0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1, \quad 0 \leqslant z \leqslant 1\,.$$ [8]

(c) In Cartesian coordinates and components, the vector field $\mathbf{G}$ is given by

$$\mathbf{G} = (4y,\ 3x,\ 2z)\,.$$

Evaluate $\int_S (\boldsymbol{\nabla}\times\mathbf{G})\cdot d\mathbf{S}$, where S is the open hemispherical surface

$$x^2 + y^2 + z^2 = r^2, \quad z \geqslant 0\,.$$ [8]

5C

(a) It is known that n people out of a population of N suffer from a certain disease, and that the other $N-n$ people do not. The test for the disease has a probability a of producing a correct positive result when used on a sufferer and a probability b of producing a false positive result when used on a non-sufferer. The test is positive when done on me. What is the probability that I am a sufferer? [9]

(b) A random variable X has density function $f(t)$ given by

$$f(t) = Ae^{-kt}, \quad \text{for} \quad t \geq 0\,,$$

where A and k are constants. Find, in terms of k:

(i) the value of A; [2]

(ii) the probability that $X \geq 3$ given that $X \geq 1$; [5]

(iii) the expectation value of X. [4]

6C Solve the equations

$$3x + 2y + 7z = a\,,$$
$$x + 4y + 6z = b\,,$$
$$y + z = c\,,$$

finding x, y and z in terms of a, b and c. [12]

Write the equations in matrix form $A\mathbf{x} = \mathbf{d}$, where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{d} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\,.$$

[2]

Hence find

$$\begin{pmatrix} 3 & 2 & 7 \\ 1 & 4 & 6 \\ 0 & 1 & 1 \end{pmatrix}^{-1}\,.$$

[6]

7D Let E be the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2 = 1\,,$$

where $a > \sqrt{2}$ and $b > \sqrt{2}$. Find the normal to the surface of E. [5]

Let S be the part of the surface of E defined by

$$0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1, \quad \text{and } z > 0,$$

and let $\mathbf{F}$ be the vector field defined by $\mathbf{F} = (-y, x, 0)$. Explain why $\int_S \mathbf{F} \cdot d\mathbf{S} = 0$ in the case $a = b$. [5]

Given that the surface area element of S is given by

$$d\mathbf{S} = \left(\frac{x}{a^2 z}\,, \frac{y}{b^2 z}\,, 1\right) dx\,dy\,,$$

find $\int_S \mathbf{F} \cdot d\mathbf{S}$ in the case $a \neq b$. [10]

8D*

(a) The $n \times n$ matrix P has components $P_{ij} = \delta_{ij} - a_i a_j - b_i b_j$ where $n > 1$ and a_i and b_i are the components of two orthogonal unit vectors $\mathbf{a}$ and $\mathbf{b}$, respectively.

(i) Evaluate $\mathrm{tr}(P)$.

(ii) Using the summation convention, or otherwise, show that $P^2 = P$.

(iii) Evaluate $P\mathbf{a}$ and hence find $\det(P)$. [12]

(b) Let S be a symmetric $n \times n$ matrix and let A be an antisymmetric $n \times n$ matrix.

(i) Show that S^2 is symmetric.

(ii) Show that $\mathrm{tr}(SA) = 0$. [8]

9E

(a) The internal energy U of a gas can be regarded as a function of the entropy S and the volume V. It is given that

$$dU = TdS - pdV\,,$$

where T and p denote temperature and pressure, respectively. The enthalpy H is defined by

$$H = U + pV\,.$$

Show that

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V$$

and also that

$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p\,.$$

[6]

(b) The heat capacities of a gas at constant pressure, C_p, and constant volume, C_V, are defined by

$$C_p = \left(\frac{\partial H}{\partial T}\right)_p\,, \quad C_V = \left(\frac{\partial U}{\partial T}\right)_V\,.$$

Using the definition of H above, show that

$$C_p - C_V = \left(\frac{\partial U}{\partial T}\right)_p + p\left(\frac{\partial V}{\partial T}\right)_p - \left(\frac{\partial U}{\partial T}\right)_V\,.$$

By considering U as a function of T and V, show that

$$\left(\frac{\partial U}{\partial T}\right)_p = \left(\frac{\partial U}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_p + \left(\frac{\partial U}{\partial T}\right)_V$$

and hence that

$$C_p - C_V = \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right]\left(\frac{\partial V}{\partial T}\right)_p\,.$$

Evaluate $C_p - C_V$ for an ideal gas, for which

$$\left(\frac{\partial U}{\partial V}\right)_T = 0 \quad \text{and} \quad pV = NkT,$$

where N is the number of gas molecules present in volume V and k is Boltzmann's constant. [14]

10E

(a) The distance $r(x,y)$ of the point (x,y) from the origin of two-dimensional Cartesian space is given by

$$r(x,y) = \sqrt{x^2+y^2}\,.$$

Obtain formulae for

$$\frac{\partial r}{\partial x}, \quad \frac{\partial r}{\partial y}$$

and also for

$$\frac{\partial^2 r}{\partial x^2}\,, \quad \frac{\partial^2 r}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 r}{\partial x\,\partial y}\,.$$

Without further calculation, write down a formula for

$$\frac{\partial^2 r}{\partial y\,\partial x}\,.$$

[8]

(b) Let $g(x,y)$ be a function defined on the xy-plane. The coordinates u and v are defined by

$$u = x\cos\theta + y\sin\theta,$$
$$v = -x\sin\theta + y\cos\theta,$$

where θ is a constant. Show that

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2}\,.$$

[12]

11F Consider the differential equation

$$\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + y = 2x\sin x\,.$$

Find a particular solution, of the form

$$y(x) = (a+bx)\sin x + (c+dx)\cos x,$$

where the constants a, b, c, and d are to be determined. [12]

Hence find the solution $y(x)$ that satisfies the initial conditions $y = 0$ and $dy/dx = 0$ at $x = 0$. [8]

12F* Consider the stationary points of the function

$$u(x, y) = 2(x-y)^2 + (x+y)^2,$$

subject to the constraint $v(x, y) = 0$, where

$$v(x, y) = x^2 - y - \tfrac{1}{4}\,.$$

By considering the function $(u - \lambda v)$, where λ is a Lagrange multiplier, show that any stationary points (x, y) satisfy:

$$x = \frac{\lambda}{(6\lambda - 16)} \quad , \quad y = \frac{\lambda(3-\lambda)}{(6\lambda - 16)}\,.$$

[8]

Deduce that λ obeys the cubic equation

$$3\lambda^3 - 21\lambda^2 + 48\lambda - 32 = 0\,.$$

[6]

You may assume that this cubic equation has only one real root (so that there is a unique stationary point). Sketch the contours of u in relation to the curve C given by $v = 0$, and explain why the function u takes its minimum value on C at this stationary point. [6]