UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Wednesday 12 June 2002 9 to 12

MATHEMATICS (2)

Before you begin read these instructions carefully:

You may submit answers to no more than ***six*** *questions. All questions carry the same number of marks.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

For ***each question*** *you have attempted, attach a* ***blue*** *cover sheet to your answer and write the question number and letter (for example,* ***3B****) in the 'section' box on the cover sheet.*

List all the questions you attempted on the ***yellow*** *master cover sheet.*

Every cover sheet must bear your candidate number and your desk number.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1A

(a) Prove that if the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar, then $\mathbf{a}$, $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$ are non-coplanar. [8]

(b) Solve the vector equation

$$\mathbf{a} \times \mathbf{r} + \lambda \mathbf{r} = \mathbf{c}$$

for $\mathbf{r}$, where $\lambda \neq 0$. [12]

2A*

(a) Calculate the derivative with respect to a of

$$\int_0^{a^2} \frac{\sin ax}{x} dx \ .$$

[8]

(b) Evaluate the integral

$$\int_0^{\infty} \cos \alpha x \, e^{-\beta x} dx \qquad (\beta > 0)$$

by writing $\cos \alpha x = \mathrm{Re}(e^{i\alpha x})$, or otherwise. [4]

Hence evaluate

$$\int_0^{\infty} x \, \cos \alpha x \, e^{-\beta x} dx \qquad (\beta > 0)$$

using the method of differentiation with respect to a parameter. [8]

3B Express the Cartesian coordinates x, y, z in terms of spherical polar coordinates r, θ, ϕ. Write down the standard volume element in spherical polar coordinates. [4]

(a) Fluid is contained within a sphere of radius a and centre the origin. The density of the fluid is $\rho = \mu(2 + (z/r))$ where μ is constant. Calculate the total mass of fluid. [6]

(b) A distribution of electric charge has charge density (i.e., charge per unit volume) $\rho = \lambda xy$ with λ a constant. It occupies the region of space with $r \leqslant a$ and $x, y, z \geqslant 0$. Calculate the total charge. [10]

4B Consider n independent events, each with two possible outcomes, one called 'success', which occurs with probability p, and the other called 'failure', which occurs with probability $q = 1 - p$. Write down the probability p_r that exactly r of the n events are successes and show that the sum of these probabilities for $0 \leqslant r \leqslant n$ is equal to one. [6]

Under certain conditions, with n large, the discrete distribution above can be approximated by a normal distribution having the same mean and variance. The approximation is $p_r \approx P(r - \frac{1}{2} \leqslant x \leqslant r + \frac{1}{2})$, where

$$P(\alpha \leqslant x \leqslant \beta) = (2\pi\sigma^2)^{-\frac{1}{2}} \int_\alpha^\beta \exp[-(x-\mu)^2/2\sigma^2]\, dx \ .$$

Write down expressions for μ and σ in terms of n, p and q. [3]

A student sits a multiple choice exam and guesses the answer to each question randomly from a selection of 4 possible answers. If the total number of questions is 60, what is the expected number of correct answers? Show, using the normal approximation above, that there is a probability greater than $\frac{1}{2}$ that the number of correct answers will lie in the range 13 to 17 inclusive. [11]

[You may assume $(2\pi)^{-\frac{1}{2}} \int_0^{\sqrt{5}/3} \exp(-\frac{1}{2}y^2)\, dy > \frac{1}{4}$.]

5C Express the system of linear equations

$$\begin{aligned} x + y + 2z &= 1 \\ x + \lambda z &= \mu \\ x + \lambda y + 2z &= 2 \end{aligned} \qquad (*)$$

in the form $A\mathbf{x} = \mathbf{b}$ by specifiying a 3×3 matrix A and column vectors $\mathbf{x}$ and $\mathbf{b}$ (λ and μ are real numbers). Determine the values of λ for which $(*)$ has a unique solution. [8]

Find all possible solutions of $(*)$ in each of the cases $\lambda = 0, 1, 2$, stating clearly any conditions on μ that may be required. [12]

6C* A vector field is defined by

$$\mathbf{A} = (\, y(z^2-1),\, x(1-z^2),\, 0\,)$$

(using Cartesian coordinates and components). Compute $\mathbf{B} = \nabla \times \mathbf{A}$ and, from this answer, compute $\nabla \cdot \mathbf{B}$. [4]

Calculate explicitly

$$\int_C \mathbf{A} \cdot d\mathbf{x}\ , \qquad \int_D \mathbf{B} \cdot d\mathbf{S}\ , \qquad \int_H \mathbf{B} \cdot d\mathbf{S}\ ,$$

where the curve C is the circle of unit radius in the xy plane with centre the origin, the surface D is the disc of unit radius in the xy plane with centre the origin, and the surface H is the hemisphere of unit radius $x^2 + y^2 + z^2 = 1$ with $z \geqslant 0$. [12]

Explain how your results illustrate (i) Stokes' Theorem; (ii) the Divergence Theorem. [4]

7D

(a) Find the most general solution of

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 6x$$

subject to $dy/dx = 0$ when $x = 0$. [10]

(b) Using the substitution $x = \cos\theta$, find the general solution of

$$\sin\theta\frac{d^2y}{d\theta^2} - \cos\theta\frac{dy}{d\theta} + 2y\sin^3\theta = 0\ .$$

[10]

8D

(a) Show that if a matrix A is symmetric and orthogonal, then $A^{-1} = A$. Let B be an orthogonal, anti-symmetric matrix. Is AB necessarily (i) anti-symmetric? (ii) orthogonal? [7]

(b) Determine the eigenvalues and normalised eigenvectors of the matrix

$$\begin{pmatrix} -1 & 0 & -3\sqrt{3} \\ 0 & 3 & 0 \\ -3\sqrt{3} & 0 & 5 \end{pmatrix}\ .$$

Verify that the eigenvectors are orthogonal. [13]

9E Write down the Taylor expansion of $f(x)$ about the point $x = a$. [2]

Find, by any method, the first three non-zero terms in the Taylor expansions about $x = 0$ of the following functions:

(i) $\frac{x}{1-e^{-x}}$; [5]
(ii) $\tan(x + \pi/4)$; [5]
(iii) $\ln \sec x$. [8]

10E* State the comparison and ratio tests for the convergence of a series. [5]

Determine for which real values of α the following series are convergent:

(i) $\sum_{n=1}^{\infty} \frac{\cos \alpha n}{n^2}$; [3]
(ii) $\sum_{n=1}^{\infty} \left(\frac{2\alpha}{\alpha^2+1}\right)^n$; [4]
(iii) $\sum_{n=1}^{\infty} n^2 e^{-\alpha n}$; [3]
(iv) $\sum_{n=1}^{\infty} [(n^4 + \alpha^4)^{1/2} - n^2]$. [5]

11F Consider the change of variables

$$x = e^{-s} \sin t \ , \quad y = e^{-s} \cos t \quad \text{such that} \quad u(x,y) = v(s,t) \ .$$

(a) Use the chain rule to express $\partial v/\partial s$ and $\partial v/\partial t$ in terms of x, y, $\partial u/\partial x$ and $\partial u/\partial y$. [6]

(b) Find, similarly, an expression for $\partial^2 v/\partial t^2$. [6]

(c) Hence transform the equation

$$y^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} = 0$$

into a partial differential equation for v. [8]

12F

(a) Consider a function $f(x,y)$. Sketch the contours of constant f in the vicinity of (i) a maximum; (ii) a minimum; (iii) a saddle point. [5]

(b) Find all stationary points of the function

$$f(x,y) = x^4 - y^4 - x^2 + 4y^2$$

and classify each as a maximum, minimum or saddle point. Draw a diagram showing the positions of all stationary points in the xy plane and sketch contours on which $f(x,y)$ is constant. [15]