UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 13 June, 2011 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

You may submit answers to ***all*** *of section A, and to no more than* ***five*** *questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

Write on one side of the paper only and begin each answer on a separate sheet. *(For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

Tie up ***all of your section A answer*** *in a single bundle, with a completed blue cover sheet.*

Each section B question has a number and a letter (for example, ***11S****). Answers to each question must be tied up in* ***separate*** *bundles and marked (for example* ***11S, 12X*** *etc) according to the number and letter affixed to each question.* ***Do not join the bundles together****. For each bundle, a blue cover sheet* ***must*** *be completed and attached to the bundle, with the correct number and letter written in the section box.*

A ***separate*** *green master cover sheet listing all the questions attempted* ***must*** *also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Green master cover sheet
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1

Evaluate the following integral

$$\int_0^{\pi/2} \sin x \; dx \, .$$

[1]

2

Evaluate the indefinite integral

$$\int \frac{1}{3-2x} \; dx \, .$$

[1]

3

(a) For $g(x) = \cosh^3 x$, find a stationary point x_1 and $g(x_1)$. [2]

(b) The function $f(x) = x^{1/x}$ is defined for $x > 0$. Find a stationary point x_0 of $f(x)$ and find $f(x_0)$. [2]

4

The position of a seven legged spider at time t is given by the equations

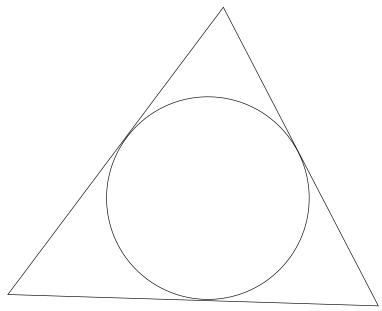
$$x = (4\pi - t)\sin t, \;\; y = (4\pi - t)\cos t, \;\; 0 \leqslant t \leqslant 4\pi$$

(a) Sketch the spider's path in the (x, y) plane. [2]

(b) Calculate the slope of the path dy/dx at the point $P(x, y) = (7\pi/2, 0)$. [2]

5

A circle is inscribed within an equilateral triangle as shown below. Find the ratio of the area of the circle to the area of the triangle:



[2]

6

Given that $\theta = \pi/2$ is a solution of

$$\sin^3\theta - \frac{1}{4}\sin\theta = \sin^2\theta - \frac{1}{4},$$

find all of the other solutions for θ explicitly, in the range $-\pi \leqslant \theta \leqslant \pi$. [2]

7

Find all real negative solutions for x of the equation

$$\sin(\pi e^x) = \frac{1}{\sqrt{2}}.$$

[2]

8

Differentiate with respect to x

$$\int_0^x e^{-y^2} \sin y \, dy\,.$$

[1]

9

Differentiate $\ln(\ln x)$ with respect to x. [1]

10

Evaluate

(a)

$$\sum_{n=-100}^{99} (n+1)^3\,,$$

[1]

(b)

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{3k}\,.$$

[1]

SECTION B

11S

(a) Write down the first four terms of the Taylor expansion of a function $f(x)$ about $x = a$. [4]

(b) Find, by any method, the Taylor expansion about $x = 0$, up to and including the term in x^3, of the following functions:

(i)
$$\frac{1}{(x^2+9)^{1/2}}\,,$$
[6]

(ii)
$$\ln[(2+x)^3]\,,$$
[4]

(iii)
$$e^{\sin x}\,.$$
[6]

12X

The Dean of Porterhouse (the oldest and most famous of the colleges of the University of Carrbridge) leads a service in Chapel every Sunday. The length s of each service, in minutes, is exponentially distributed with a mean of $\bar{s}$ minutes.

(a) Write down the probability density function for s. [2]

Unfortunately, some of the Dean's services are being interrupted as a result of an electrical fault in the chapel organ. This fault causes one of the organ's pipes to spontaneously emit a loud sound t minutes after the beginning of the service. It is found that t is exponentially distributed with mean $\bar{t}$ minutes and is independent of s.

(b) Draw a pair of axes at right-angles to each other labelling one s and one t. Indicate on this diagram the region of the (s, t)-plane in which the service is **not** interrupted by the organ. [2]

(c) The probability of being in some region of this plane is the double integral of the product of the density functions for s and t integrated over the region. Explain in words why this is so. [2]

(d) Calculate the *probability* (as a function of $\bar{s}$ and $\bar{t}$) that the service is **not** interrupted by the organ. [4]

Define the random variable r to be equal to "-1" if the organ does not interrupt the service, and equal to "the number of minutes of the service which are remaining, at the moment the organ makes a noise" if the organ interrupts the service.

(e) On the same diagram as before, indicate the region of the (s, t)-plane in which $r > r_0$, where r_0 is a positive constant. [2]

(f) Calculate the *probability* (as a function of r_0, $\bar{s}$ and $\bar{t}$) that r is greater than r_0 minutes, again assuming that r_0 is a positive constant. [4]

(g) Consider the answer for $P(r > r_0)$ that you have found in (f), and comment on whether it seems sensible in each of the following limits:

(i) $r_0 \to \infty$, (ii) $r_0 \to 0$ in the case $\bar{s} = \bar{t}$. [2]

Suppose that it is 6:54pm, that the service was interrupted by the organ at 6:22pm, and that the Dean is *still* talking. The Fellows are getting hungry, and are wondering how many *more* minutes, m, they are going to have to stay sitting in Chapel until the service ends.

(h) State (or calculate) the expected value of m. [2]

13Y

(a) Consider the matrix A where

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix} .$$

(i) Compute $\det A$. [2]

(ii) Compute A^{-1}. [3]

(iii) Compute the eigenvalues and eigenvectors of A. [4]

(iv) Show that the eigenvectors are orthogonal. [1]

(b) (i) Show that, taking A from part (a),

$$(x\ y) A \begin{pmatrix} x \\ y \end{pmatrix} = 1 \qquad (*)$$

leads to the equation of an ellipse

$$bx^2 + cxy + dy^2 = 1 .$$

Find the coefficients b, c and d. [4]

(ii) Defining $\begin{pmatrix} x \\ y \end{pmatrix} = x'\mathbf{x}_1 + y'\mathbf{x}_2$ where $\mathbf{x}_1$ and $\mathbf{x}_2$ were the eigenvectors found in (a)(iii), determine the form of $(*)$ in terms of (x', y'). Suggest a linear re-scaling of coordinates to (x'', y'') such that they describe a unit circle, determining the matrix B such that

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix} .$$

[6]

14R

This question involves solving the differential equation

$$\sqrt{3}\frac{dy}{dx} + y = 4\sin x$$

by Fourier methods.

(a) Write down a Fourier series expansion of an arbitrary periodic function which has period 2π. [3]

(b) Suppose that $y(x)$ has such an expansion. Substitute the Fourier series expansion into the differential equation in order to obtain a constraint on its coefficients. [2]

(c) Why may we equate the coefficients of $\sin(mx)$ in this constraint (for each integer m)? You may also equate the coefficients of $\cos(mx)$. [2]

(d) By equating coefficients as described in (c), find all of the coefficients of the Fourier series expansion of $y(x)$. [5]

(e) Thus, write down the explicit form of the periodic solution $y(x)$ in only one term. [1]

(f) Sketch $y(x)$ for $0 \leqslant x \leqslant 2\pi$, clearly displaying maxima and minima. [3]

(g) Use Parseval's theorem to evaluate $\int_0^{2\pi}\{y(x)\}^2 dx$. [2]

(h) Check your answer to (g) by performing the integral explicitly. [2]

15S

(a) Write down an expansion of the vector $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c})$, as a linear combination of $\mathbf{b}$ and $\mathbf{c}$. [2]

(b) By considering two expressions for $(\mathbf{a} \wedge \mathbf{b}) \wedge (\mathbf{c} \wedge \mathbf{d})$, or otherwise, derive the identity

$$[\mathbf{b}, \mathbf{c}, \mathbf{d}]\mathbf{a} - [\mathbf{a}, \mathbf{c}, \mathbf{d}]\mathbf{b} + [\mathbf{a}, \mathbf{b}, \mathbf{d}]\mathbf{c} - [\mathbf{a}, \mathbf{b}, \mathbf{c}]\mathbf{d} = \mathbf{0},$$

where we define

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}).$$

[6]

(c) Show that a necessary condition for the lines

$$\mathbf{r} = \mathbf{a} + s\mathbf{m}\,, \qquad \mathbf{r} = \mathbf{b} + t\mathbf{n}$$

to intersect is

$$[(\mathbf{a} - \mathbf{b}), \mathbf{m}, \mathbf{n}] = 0.$$

[6]

(d) Find the values of s and t at the point of intersection in terms of triple products of $\mathbf{a}, \mathbf{b}, \mathbf{m}$ and $\mathbf{n}$ (assuming that the intersection occurs and that $[\mathbf{a}, \mathbf{m}, \mathbf{n}] \neq 0$). [6]

16T

(a) Solve the differential equation

$$x\,\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

such that $y = -1$ and $dy/dx = 2$ at $x = 1$. [5]

[*Hint: By a suitable change of variable transform the equation into a first order equation.*]

(b) Show that

$$2x\left(ye^{x^2} - 1\right)dx + e^{x^2}\,dy$$

is an exact differential and use this result to find the solution of

$$\frac{dy}{dx} = -2xy + 2xe^{-x^2},$$

such that $y = -1$ at $x = 0$. [5]

(c) Determine the solution of

$$\frac{d^2y}{dx^2} + 4y = \sin x + \cos x,$$

such that $y = 3$ at $x = 0$ and $y = 0$ at $x = \pi/4$. [10]

17Z

(a) Find the moduli and arguments of

(i) $z = 1 - i\sqrt{3}$, [4]

(ii) $z = e^{i\pi/2} + \sqrt{2}e^{i\pi/4}$, [4]

(iii) $z = (1 + i)e^{i\pi/6}$. [4]

(b) Find all complex solutions of the equation $z^6 + z^3 + 1 = 0$. [8]

18Z

(a) State a necessary condition for the expression

$$p(x, y)dx + q(x, y)dy$$

to be an exact differential. [2]

(b) Determine which of the following expressions are exact differentials and which are not:

(i) $2\sin x \cosh y\, dx + \sin^2 x \sinh y\, dy$,

(ii) $\sinh(x + iy)\, dx - \sin(y - ix)\, dy$. [5]

(c) By finding a suitable integrating factor (or otherwise) solve:

$$(\cos x + y \sin x)\, dx + x \sin x\, dy = 0.$$

[10]

(d) Write the following expression as a simpler single partial derivative, given that a is a function of two variables:

$$\left(\frac{\partial a}{\partial b}\right)_c \left(\frac{\partial b}{\partial d}\right)_e + \left(\frac{\partial c}{\partial d}\right)_e \left(\frac{\partial a}{\partial c}\right)_b .$$

[3]

19R*

Evaluate $\int \mathbf{F}.\mathbf{dS}$ for:

(a) Any closed surface S and $\mathbf{F} = (y^2,\ e^{xz},\ e^{xy})$. [3]

(b) S being the entire $y = 0$ plane and

$$\mathbf{F} = \left(\frac{\sin y}{y},\ (y+1)e^{-(x^2+\sin^3 y+z^2)},\ (z^2+1)^{-(x^2+1)}\right).$$

[5]

(c) $\mathbf{F} = k(x,\ y,\ z)$ where k is a constant, for the surface of a sphere S of radius r centred on the point $(\pi,\ \sqrt{2},\ e)$. [4]

(d) S being the part of $z = 5 - x^2 - y^2$ with $z \geqslant 1$ and $\mathbf{F} = \nabla \times \mathbf{A}$, where $\mathbf{A} = (yz^2,\ -3xy,\ x^3y^3)$. [8]

20X*

(a) Three functions $f_0(x)$, $f_1(x)$ and $f_2(x)$ are defined by:

$$f_n(x) = \left(\frac{x - \frac{\pi}{2}}{x}\right)^n \sin(\tan x)$$

for $n = 0$, 1 or 2, except at $x = m\pi/2$ for any integer m, where $f_n(x)$ is defined to be zero.

(i) By means of a rough sketch of $f_0(x)$ in the range $-\pi < x < +\pi$, find any places where the function is not **continuous**, or state that there are no such places. Indicate any places where the function is not **differentiable**, or state that there are no such places. [4]

(ii) Repeat (i) for $f_1(x)$. [4]

(iii) Repeat (i) for $f_2(x)$. [4]

(b) A function $g(x)$ is defined by

$$g(x) = \frac{1}{x^3 \sin(1/x^5)}.$$

Find the first two non-zero terms in the expansion for $g(x)$ valid in the limit of large x, and indicate the order of the next omitted term. [8]

END OF PAPER