UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part IA

Monday, 7 June, 2010 9:00 am to 12:00 pm

MATHEMATICS (1)

Before you begin read these instructions carefully:

The paper has two sections, A and B. Section A contains short questions and carries 20 marks in total. Section B contains ten questions, each carrying 20 marks.

*You may submit answers to **all** of section A, and to no more than **five** questions from section B.*

The approximate number of marks allocated to a part of a question is indicated in the right hand margin.

***Write on one side of the paper only and begin each answer on a separate sheet.** (For this purpose, your section A attempts should be considered as one single answer.)*

Questions marked with an asterisk (*) require a knowledge of B course material.

At the end of the examination:

*Tie up **all of your section A answer** in a single bundle, with a completed blue cover sheet.*

*Each section B question has a number and a letter (for example, **11S**). Answers to these questions must be tied up in **separate** bundles, marked **R, S, T, X, Y** or **Z** according to the letter affixed to each question. **Do not join the bundles together**. For each bundle, a blue cover sheet **must** be completed and attached to the bundle, with the correct letter **R, S, T, X, Y** or **Z** written in the section box.*

*A **separate** yellow master cover sheet listing all the questions attempted **must** also be completed. (Your section A answer may be recorded just as A: there is no need to list each individual short question.)*

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
6 blue cover sheets and treasury tags
Yellow master cover sheet
Script paper

SPECIAL REQUIREMENTS
Approved calculators allowed.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Given that $x = 1$ is a solution of

$$2x^3 - 9x^2 + 13x - 6 = 0\,,$$

find all solutions of this equation. [2]

2 Differentiate with respect to x

(a) e^{-x^2}, [1]

(b) $\sinh^2 x\,.$ [1]

3 Calculate

(a) $\displaystyle\int \ln x \; dx\,,$ [1]

(b) $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} \sin x \; dx\,.$ [1]

4 A curve is given parametrically by

$$x = e^t$$

$$y = \sin t\,.$$

Calculate its slope $\dfrac{dy}{dx}$ at $x = 1\,.$ [2]

5 Determine the radius and the position of the centre of the circle

$$x^2 + y^2 - 6x - 2y + 1 = 0\,.$$

[2]

6 For what value of x and k is the line $y = kx + 1$ tangent to the curve $y = \ln x$? [2]

7 Evaluate

(a)

$$\sum_{n=0}^{\infty} (0.999)^n \ ,$$

[1]

(b)

$$\sum_{n=0}^{N} (n+1) \ .$$

[1]

8 Sketch the graphs of

(a) $\dfrac{2}{|x-3|}$ for $-\infty < x < \infty$, [1]

(b) $\dfrac{1}{\sqrt{2}}(\sin x + \cos x)$ for $-\pi/4 \leqslant x \leqslant \pi/4$. [1]

9 Find the solutions of

$$\sqrt{2x^2+17} + x \ = \ 3\,.$$

[2]

10 Solve

$$1 + \sin 2x \ = \ 2\cos^2 x$$

for $0 \leqslant x \leqslant \pi$. [2]

SECTION B

11S

Write down the first four terms of the Taylor expansion of a function $f(x)$ about $x = x_0$. [4]

(a) Taking $f(x) = x^{1/4}$, find an approximation for $(17)^{1/4}$ as a sum of fractions using the first three terms of the Taylor expansion of $f(x)$ about $x_0 = 16$. [6]

(b) Find the expansions about $x_0 = 0$ up to and including terms in x^3 of

(i) $e^x \sin x$, [5]

(ii) $\ln(1 + x + x^2)$. [5]

12X

Porterhouse, the oldest and most famous college in the University of Carrbridge, has F Fellows. After dinner, all the fellows seat themselves around a large circular table to drink of Port and eat Stilton.

Every few minutes the Junior Fellow is required **either** to cough (which he does with probability p) **or** to sneeze (which he does therefore with probability $q = 1 - p$). If he coughs, the Port is passed one position to the left. If he sneezes, the Stilton is passed one position to the right. In this manner, the Port and Stilton make progress around the table in opposite directions.

Initially, the Port and Stilton are placed together in front of the Senior Fellow. For notational purposes, the Port is said to start "in position 0". Each time the Port moves, its position **increases** by one, and so the Port visits positions 0, 1, 2, ... in turn. Likewise, the Stilton, though starting in the same place, is said to start "in position F", and each time it moves its position **decreases** by one. It therefore visits positions F, $F-1$, $F-2$, ... in turn. Consequently, the first time the Port meets the Stilton, the position numbers of both will be identical and we can talk of them "meeting at position n" *etc.*

(a) What is the probability that the Stilton manages to complete one full circuit while the Port remains stationary? [1]

(b) Write down the probability, p_n^{meet}, that the Port and Stilton first meet at position n. (Here and in all subsequent parts of this question you may assume $0 \leqslant n \leqslant F$ and $F \geqslant 2$.) [3]

(c) Write down the mean position, μ, at which the Port and Stilton first meet. Write down also the variance σ^2 of the position of first meeting. [3]

(d) Calculate the probability, $p_n^{\text{specialMeet}}$, that the Port and Stilton first meet at position n with the Port having been first to arrive there. [6]

(e) Calculate the probability that the Port reaches position n *before* the Stilton gets there, leaving your answer as a sum of terms of the form p_n^{meet} and $p_n^{\text{specialMeet}}$ for suitable values of n. Explain your reasoning. [4]

(f) Hence (or otherwise) prove the identity

$$\sum_{k=0}^{F-n-1} \binom{n-1+k}{k} p^n q^k = \binom{F-1}{n} p^n q^{F-n} + \sum_{k=n+1}^{F} \binom{F}{k} p^k q^{F-k},$$

being careful to explain your reasoning. [3]

13Y

(a) (i) Find the eigenvalues and eigenvectors of the matrix (show your working)

$$M = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 4 & -2 \\ -1 & -2 & 2 \end{pmatrix}.$$

[7]

(ii) Show that the eigenvectors of matrix M are mutually orthogonal. [3]

(b) Consider the following set of linear simultaneous equations

$$\begin{aligned} x + 2\,y + 3\,z &= 1 \\ 3\,x + 4\,y + 5\,z &= 2 \\ x + 3\,y + 4\,z &= 3\,. \end{aligned}$$

(i) Using matrix methods show that these simultaneous equations have a unique solution. [4]

(ii) Find the solution, again using matrix methods. [6]

14Z

(a) Find the real and imaginary parts of

(i) $\dfrac{2+3i}{3+2i}$, [2]

(ii) $\ln\left[\dfrac{1}{2}\left(\sqrt{3}+i\right)\right]$, [5]

(iii) i^{-5}, [4]

(iv) $\left(-\dfrac{1}{2}+i\,\dfrac{\sqrt{3}}{2}\right)^2$. [4]

(b) Find all solutions of the equation

$$z^4 - 1 = i\sqrt{3}\,.$$

[5]

15S

Given the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$,

(a) The vector $\mathbf{x}$ satisfies the equation

$$\mathbf{x} = \mathbf{a} - (\mathbf{b}\cdot\mathbf{x})\mathbf{b} - \mathbf{b}\wedge\mathbf{x}\,.$$

Show that

$$\mathbf{x}\cdot\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{1+|\mathbf{b}|^2}$$

and thus find the solution for $\mathbf{x}$. [10]

(b) Show that the equation

$$\mathbf{x} = \mathbf{a} + (\mathbf{b}\cdot\mathbf{x})\mathbf{c}$$

has a unique solution provided that $\mathbf{b}\cdot\mathbf{c} \neq 1$. Discuss carefully the case $\mathbf{b}\cdot\mathbf{c} = 1$. [10]

16T

Let

$$\Gamma(p) \;=\; \int_0^\infty x^{\,p-1} e^{-x}\, dx\,,$$

$$B(p,q) \;=\; \int_0^1 x^{\,p-1}(1-x)^{q-1}\, dx\,,$$

be the definitions of the gamma and beta functions for positive p and q.

(a) Show that $B(q,p) = B(p,q)$. [4]

(b) Use $x = \sin^2\theta$ to obtain the trigonometric form of the beta function

$$B(p,q) \;=\; 2\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\, d\theta\,.$$

[4]

(c) Show that

$$B(n,n) \;=\; \frac{B\left(n,\frac{1}{2}\right)}{2^{2n-1}}\,.$$

[6]

(d) Show that

$$\Gamma(p)\,\Gamma(q) \;=\; B(p,q)\,\Gamma(p+q)\,.$$

[6]

[Hint: write the left hand side as a multiple integral and consider possible changes of variables.]

17T

(a) Find the substitution that simplifies the differential equation

$$x\,\frac{dy}{dx} + y = e^{xy}\,,$$

and give its solution for $y = 1$ at $x = 1$. [6]

(b) Solve the differential equation

$$(1 + x^3)\,\frac{dy}{dx} = x^2 y\,,$$

such that $y = 2$ at $x = 1$. [6]

(c) Find the general solution of

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0$$

[4]

and the solution of

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = x\,,$$

such that $y = 1$ and $dy/dx = 2$ at $x = 1$. [4]

18X

A function u of two variables x and t is defined by

$$u(x,t) \;=\; t^a y\,(b^a t^a x) \qquad (*)$$

where a and b are (non-zero) real constants, and y is an arbitrary function of a single variable.

(a) Write down expressions for the following derivatives in terms of y

(i)

$$\frac{\partial u}{\partial t}\,,$$

[3]

(ii)

$$\frac{\partial^3 u}{\partial x^3}\,.$$

[3]

(b) Now assume that the (previously arbitrary) function $y(s)$ satisfies the homogeneous Airy equation

$$\frac{d^2 y}{ds^2} - sy \;=\; 0.$$

Use this fact to prove that the function $u(x,t)$ in $(*)$ satisfies the reduced Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} \;=\; 0$$

for a suitable choice of the (non-zero) real-valued constants a and b, whose values you should find. [14]

19Z*

(a) Give a necessary condition for

$$P(x, y)\, dx + Q(x, y)\, dy$$

to be an exact differential. [2]

(b) Find an expression for the change in

$$f(x, y) \;=\; x\, e^{xy}$$

to first order in δx and δy, when the values of x and y are simultaneously changed from 1 to $1 + \delta x$ and from 1 to $1 + \delta y$. [3]

(c) Use the method of separation of variables to find the most general solution $u(x, y)$ of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \;=\; -25u$$

given that u satisfies the boundary conditions

$$u(x, 0) \;=\; u(x, \pi) \;=\; u(0, y) \;=\; u(\pi, y) \;=\; 0\,.$$

[15]

20Y*

(a) Determine whether the following series are convergent

(i)

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}\,,$$

[3]

(ii)

$$\sum_{n=1}^{\infty} \frac{n^3}{(\ln 2)^n}\,.$$

[3]

(b) For which values of x is the series below absolutely convergent?

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

[4]

(c) Evaluate the following limits

(i)

$$\lim_{x \to 0} \frac{\cos x - \cos 3x}{x^2}\,,$$

[5]

(ii)

$$\lim_{x \to 0} \frac{\ln \cos x}{\ln \cos 3x}\,.$$

[5]

END OF PAPER