UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Monday 31 May, 2004 1.30 to 4.30

STOCHASTIC CALCULUS AND APPLICATIONS

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathbb{X} = (\Omega, \mathcal{F})$ be a measurable space equipped with a probability measure $\mathbb{P}$ and a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and let $X_t(\omega)$ be a continuous $(\mathcal{F}_t, \mathbb{P})$-semimartingale.

a) Show that a continuous local martingale of finite variation starting from 0 must necessarily be identically 0, $\mathbb{P}$-a.s.; therefore conclude that the decomposition of X_t into a local martingale part and a part of finite variation is unique.

b) Show that the quadratic variation process $[X]_t$ does not depend on the filtration. (You may use without proof any formula for $[X]_t$ proved in the lectures). Suppose now that $\mathbb{Q}$ is another probability measure, absolutely continuous with respect to $\mathbb{P}$ and that X is a $(\mathbb{Q}, \mathcal{F}_t)$-semimartingale, as well. Show that the quadratic variation process of X is the same regardless of which measure ($\mathbb{P}$ or $\mathbb{Q}$) we use to compute it.

c) Prove that a local martingale M_t is a martingale if and only if for all $t > 0$ the family:

$$\{M_T : T \text{ is a stopping time } \quad \leq t\}$$

is uniformly integrable.

2 a) State and prove the integration by parts formula for continuous semimartingales. State (the multidimensional) Itô's formula, and describe (without proof) how to establish it from the integration by parts formula.

b) Let $X(t) = B(t) + \mu t$ $(\mu \neq 0)$, where $B(t)$ is a Brownian motion on $\mathbb{R}$ started from x. For $b > |x|$, set $T_+ = \inf\{t \geq 0 : X(t) \geq b\}$, $T_- = \inf\{t \geq 0 : X(t) \leq -b\}$ and $T = T_- \wedge T_+$. First show that $\mathbb{E}[T^2] < \infty$. Then, using a suitable function f such that $f(X_t)$ is a martingale, compute the probability $\mathbb{P}(T_- < T_+)$.

3 Let $B(t)$ be standard Brownian motion on $\mathbb{R}$, and $\mathcal{F}_t$ be its natural filtration. For a $\lambda > 0$, we define:

$$X(t) = B(e^{2\lambda t}), \text{ and } \mathcal{G}_t = \mathcal{F}_{e^{2\lambda t}}.$$

a) Show that $X(t)$ is a $\mathcal{G}_t$-martingale and compute its quadratic variation process.

b) Write $X(t)$ as a stochastic integral with respect to a $\mathcal{G}_t$-adapted Brownian motion $W(t)$ (that you will construct).

c) Conclude that the process

$$Y(t) = e^{-\lambda t} B(e^{2\lambda t})$$

satisfies the Ornstein-Uhlenbeck stochastic differential equation:

$$dY(t) = \sqrt{2\lambda} dW(t) - \lambda Y(t)\, dt,$$

and solve the equation to express $Y(t)$ explicitly in terms of $W(\cdot)$.

d) Suppose $Z(t)$ is a *stationary* process satisfying $E[Z(0)^2] = 1$, and the stochastic differential equation:

$$dZ(t) = \sqrt{2\lambda} dW(t) + g(t, \omega)\, dt,$$

where g is a $\mathcal{G}_t$ adapted process such that $E[g^2(t, \omega)] \leqslant \lambda^2$, for all t. By considering the semimartingale decomposition of $Z(t)^2$, or otherwise, prove that $Z(\cdot) = Y(\cdot)$, a.s..

4 a) The following stochastic differential equation:

$$dX(t) = \alpha X(t)\, dB(t) + X^{\delta}(t)\, dt$$
$$X(0) = x_0 > 0,$$

where $\alpha \in \mathbb{R}$ and $\delta > 0$, has as you know a unique maximal local solution on $(0, \infty)$. Write down the equation satisfied by $Y(t) = X(t)/F(B(t), t)$ where $F(B(t), t) = \exp(\alpha B(t) - \alpha^2 t/2)$, and solve it to find the solution to the original equation explicitly.

b) For which values of δ does the solution exist for all $t \geq 0$ with probability 1?

5 a) Let $\mathbb{X}$ be a measurable space equipped with a filtration $\mathcal{F}_t$, and let $\mathbb{P}$, $\mathbb{Q}$ be two probability measures on $\mathbb{X}$ such that for $A \in \mathcal{F}_t$, we have:

$$\mathbb{Q}(A) = \int_A \exp(M_t - [M]_t/2) \, d\mathbb{P},$$

where the process M is a continuous $(\mathcal{F}_t, \mathbb{P})$-martingale. Show that if X is a continuous $(\mathcal{F}_t, \mathbb{P})$-(local martingale) then $\tilde{X} := X - [X, M]$ is a continuous $(\mathcal{F}_t, \mathbb{Q})$-(local martingale).

b) Show that if X is a Brownian motion under $\mathbb{P}$, then $\tilde{X}$ is a Brownian motion under $\mathbb{Q}$. Use this fact to show that if $\mathbb{P}^x$ is the distribution of the solution to the stochastic differential equation:

$$dX_s = dB_s - \mu X_s \, ds, \qquad X_0 = x,$$

and $\mathbb{Q}^x$ is the Wiener measure on paths $(\omega_s : s \geqslant 0)$ in $C([0, \infty); \mathbb{R})$ starting from x, then:

$$\left.\frac{d\mathbb{P}^x}{d\mathbb{Q}^x}\right|_{\mathcal{F}_t} = \exp\Big(-\mu \int_0^t \omega_s \, d\omega_s - \tfrac{\mu^2}{2} \int_0^t \omega_s^2 \, ds\Big). \tag{1}$$

6 Let $u(t, x)$ be a function in $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}_+)$ satisfying the following partial differential equation problem:

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) &= \frac{1}{2}\Delta u(t, x) - \lambda x \cdot \nabla u(t, x) + \frac{\lambda^2 |x|^2}{2} u(t, x) \\ u(0, x) &= 1. \end{cases}$$

a) By formally applying the Feynman-Kac formula, or otherwise, give a probabilistic interpretation of the solution $u(t, x)$ as an integral on the space of continuous paths.

b) Write $u(t, x)$ as an integral on the space of continuous paths with respect to the Wiener measure. You may find (1) useful, and you may use it.

c) Now find $u(t, x)$ explicitly.