UNIVERSITY OF CAMBRIDGE

M. Phil. in STATISTICAL SCIENCE

Thursday 3 June, 2004 9 to 11

MATHEMATICAL METHODS IN GENERAL EQUILIBRIUM THEORY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $x : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L$ be a (nonempty-valued) demand correspondence.

(a) Define the indirectly revealed at-least-as-preferable-as relation derived from x.

(b) What does it mean for x to satisfy the congruence axiom?

(c) Denote by $\succsim$ the indirectly revealed at-least-as-preferable-as relation derived from x and suppose that x satisfies the congruence axiom. Let $\mathcal{P}$ be the set of the complete and transitive binary relations $\succsim'$ having the following properties: For every $y \in \mathbb{R}^L$ and every $z \in \mathbb{R}^L$, if $y \succsim z$, then $y \succsim' z$; and if $y \succ z$, then $y \succ' z$. Prove then that the intersection of all binary relations in $\mathcal{P}$, when viewed as subsets of $\mathbb{R}^L \times \mathbb{R}^L$, coincides with $\succsim$.

2 Consider an exchange economy with L goods and an atomless probability measure space $(A, \mathcal{A}, \mu)$ of consumers. Suppose that the consumption sets of all consumers are $\mathbb{R}^L_+$.

(a) Give the definition of a Pareto efficient allocation in this economy.

(b) Give the definition of a price quasi-equilibrium in this economy.

(c) Suppose in addition that all consumers have complete, transitive, continuous, but not necessarily convex preference relations. Prove that for every Pareto efficient allocation, there is a non-zero price vector such that the two constitute a price quasi-equilibrium. (*Hint*: Consider the convex hull of the set

$$\bigcup_{a \in D} \{z \in \mathbb{Q}^L \mid f(a) + z \succ_a f(a)\},$$

where $D \in \mathcal{A}$ and $\mu(D) = 1$, and $\mathbb{Q}^L$ is the set of all vectors with rational coordinates. Then apply the Lyapunov theorem and the supporting hyperplane theorem.)

3 Let $P = \{p \in \mathbb{R}^L_+ \mid p^1 + \cdots + p^L = 1\}$; Z be a nonempty, convex, and compact subset of $\mathbb{R}^L$; and $z : P \to Z$ be a correspondence such that $z(p)$ is nonempty and convex for every $p \in P$, the graph $\{(p, v) \in P \times Z \mid v \in z(p)\}$ is a closed subset of $P \times Z$, and $p \cdot v = 0$ for every (p, v) with $v \in z(p)$.

(a) Define a correspondence $g : Z \to P$ by letting $g(v) = \{p \in P \mid p \cdot v \geqslant q \cdot v \text{ for every } q \in P\}$. Then define a correspondence $f : P \times Z \to P \times Z$ by $f(p, v) = g(v) \times z(p)$. Prove that f satisfies the assumptions of Kakutani's fixed-point theorem.

(b) Prove that there exists a $p \in P$ such that $z(p) \cap (-\mathbb{R}^L_+) \neq \varnothing$, where $-\mathbb{R}^L_+ = \{v \in \mathbb{R}^L \mid v^\ell \leqslant 0 \text{ for every } \ell\}$.

(c) Suppose in addition that for every $p \in P$, if $p^\ell = 0$ for some ℓ, then for every $v \in z(p)$, there exists a k such that $v^k > 0$. (The two coordinates ℓ and k may actually be the same.) Prove then that there exists a $p \in P$ such that $0 \in z(p)$.

4 Consider a class of exchange economies with two goods and two consumers. The two consumers' preference relations, the second consumer's initial endowment vector, and the first consumer's initial endowment for the first good are fixed. The economies are therefore parameterized by the first consumer's initial endowment for the second good, which we denote by $\omega_{21} \in \mathbb{R}_{++}$. Assume that the excess demand functions are infinitely many times differentiable.

(a)Under what condition in terms of the parameterized aggregate excess demand function is it guaranteed that almost every economy is regular?

(b) If the first consumer's preference relation can be represented by the utility function $u(x_{11}, x_{21}) = \ln x_{11} + x_{21}$, where x_{11} must be positive but x_{21} may be negative, is the condition you identified in (a) always satisfied?

(c) If, in addition, the second consumer has the same preference relation as the first consumer, is the condition you identified in (a) always satisfied?

5 Denote by $\mathcal{P}$ the set of all complete, transitive, and continuous preference relations on $\mathbb{R}^L_+$.

(a) Give the definition of the closed convergence topology on $\mathcal{P}$.

(b) Let $\succsim \in \mathcal{P}$ and $(\succsim_n)_n$ be a sequence in $\mathcal{P}$. State, without proof, an equivalent set of conditions for $\succsim_n \to \succsim$ as $n \to \infty$ with respect to the closed convergence topology.

(c) Suppose that $\succsim$ can be represented by a continuous utility function $u : \mathbb{R}^L_+ \to \mathbb{R}$ and, for every n, $\succsim_n$ can be represented by a continuous utility function $u_n : \mathbb{R}^L_+ \to \mathbb{R}$. Suppose also that for every $k > 0$ and every $\varepsilon > 0$, there exists an $N > 0$ such that

$$\|u_n(x) - u(x)\| < \varepsilon$$

for every $n > N$ and every $x \in \mathbb{R}^L_+$ with $\|x\| \leqslant k$. Is it then true that $\succsim_n \to \succsim$ as $n \to \infty$ with respect to the closed convergence topology? In particular, are the conditions you identified in (b) always satisfied?