UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Friday 30 May 2008 9.00 to 12.00

ADVANCED PROBABILITY

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the term σ-field. Let $(\mathcal{F}_r : r \in R)$ be a collection of σ-fields of subsets of Ω. Show that $\cap_r \mathcal{F}_r$ is a σ-field, but that $\cup_r \mathcal{F}_r$ need not be. Show that there exists a smallest σ-field of Ω containing every $\mathcal{F}_r$.

Let $X_1, X_2, \ldots$ be independent random variables on $(\Omega, \mathcal{F}, P)$. Define the tail σ-field of the X_i, and show that every event in the tail σ-field has probability either 0 or 1.

Let $S_n = X_1 + X_2 + \ldots + X_n$. Show that the events

$$\left\{\liminf_{n\to\infty} S_n/\sqrt{n} \leqslant -x\right\}, \quad \left\{\limsup_{n\to\infty} S_n/\sqrt{n} \geqslant x\right\}$$

lie in the tail σ-field for every $x \in \mathbb{R}$.

Suppose further that the X_i are symmetric (in that X_i and $-X_i$ have the same distribution), and that there exists $c \in \mathbb{R}$ such that $P(|X_i| \leqslant c) = 1$ for all i. Show that

$$P\left(|S_n| \leqslant \frac{1}{2}c \quad \text{infinitely often}\right) = 1\,.$$

2 (a) State and prove the two Borel–Cantelli lemmas.

(b) Show that the $N(0,1)$ distribution function Φ and density function ϕ satisfy

$$1 - \Phi(x) \sim \frac{\phi(x)}{x} \quad \text{as} \quad x \to \infty\,.$$

(c) Let $X_1, X_2, \ldots$ be independent $N(0,1)$ random variables on $(\Omega, \mathcal{F}, P)$. Show that

$$P\left(\limsup_{n\to\infty} \frac{X_n^2}{\log n} = 2\right) = 1\,.$$

3 Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) Define the space $L^2 = L^2(\Omega, \mathcal{F}, P)$ and show that it is complete in that, for any Cauchy sequence $X_n \in L^2$ there exists $X \in L^2$ such that $X_n \to X$ in L^2.

(b) Define the terms 'filtration' and 'martingale'. State the almost-sure martingale convergence theorem.

(c) Let $(X_n : n \geqslant 0)$ be a martingale such that $E(X_n^2) \leqslant M < \infty$ for all n. Show that

$$E\left((X_n - X_m)^2\right) = E\left(X_n^2\right) - E\left(X_m^2\right)$$

for $m \leqslant n$, and deduce that X_n converges in L^2.

4 (a) Let $X_1, X_2, \ldots$ be independent integrable random variables with $E(X_i) = 0$ for all i. Show that $S_n = X_1 + X_2 + \ldots + X_n$ defines a martingale with respect to the filtration given by

$$\mathcal{F}_n = \sigma\left(X_1, X_2, \ldots, X_n\right) .$$

(b) Suppose further that the X_i are identically distributed and take values in $\{\ldots, -2, -1, 0, 1\}$ with $P(X_1 = 1) > 0$. Let $M(t) = E(e^{tX_1})$ and suppose $\tau > 0$. Show that $M(\tau) \in [1, \infty)$, and that

$$Z_n = e^{\tau S_n} / M(\tau)^n$$

is a martingale.

(c) [Continuation] Let $b \in \{1, 2, \ldots\}$ and $T = \inf\{n : S_n = b\}$. By considering the non-negative martingale $b - S_{n \wedge T}$ or otherwise, show that $P(T < \infty) = 1$.

(d) [Continuation] Show that the martingale $Z_{n \wedge T}$ is uniformly integrable, and deduce that

$$E\left(M(\tau)^{-T}\right) = e^{-\tau b} .$$

(e) Calculate $E(e^{-\alpha T})$ for $\alpha > 0$, in the special case when

$$P\left(X_1 = 1\right) = P\left(X_1 = -1\right) = \frac{1}{2} .$$

[Any general result which you use should be stated clearly.]

5 (a) Define the term 'uniformly integrable'. Let Z be integrable and $\mathcal{F}_n$ a filtration. Show that the sequence $X_n = E(Z|\mathcal{F}_n), n \geqslant 1$, is uniformly integrable.

(b) For $x \in [0,1)$, define for non-negative integers k, n,

$$b_n(x) = k2^{-n} \quad \text{if} \quad k2^{-n} \leqslant x < (k+1)2^{-n} \,.$$

Let $f : [0,1] \to \mathbb{R}$ be integrable, and let U be uniformly distributed on $[0,1]$. Show that $X_n = E(f(U)|\mathcal{F}_n)$ defines a uniformly integrable martingale with respect to the filtration $\mathcal{F}_n = \sigma\left(b_n(U)\right)$. Let f_n be the step function on $[0,1]$ given by

$$f_n(x) = 2^n \int_{b_n(x)}^{b_n(x)+2^{-n}} f(u)du \,.$$

Show that $f_n(x) \to f(x)$ for almost every x, and

$$\int_0^1 |f_n(u) - f(u)|du \to 0 \quad \text{as} \quad n \to \infty \,.$$

6 Define a standard Brownian motion $B = (B_t : t \geqslant 0)$. Give a careful statement of the Strong Markov Property. Set

$$M_t = \sup\left\{B_s : 0 \leqslant s \leqslant t\right\} \,.$$

Prove that

$$P\left(M_t \geqslant m,\ B_t \leqslant x\right) = P\left(B_t \geqslant 2m - x\right)$$

for $t \geqslant 0$, $m > 0$, and $x \leqslant m$.

Deduce that M_t has the same law as $|B_t|$.

For $x > 0$, let $T_x = \inf\{t : B_t \geqslant x\}$. Show that T_x has the same law as $(x/B_1)^2$, and calculate the density function of T_x.

END OF PAPER