UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Monday 6 June, 2005 1:30 to 4:30

ADVANCED FINANCIAL MODELS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on optimal hedging in the least-squares sense in a one-period financial model.

Your essay should cover the notions of attainable claims, dominated and equivalent martingale measures, the minimal martingale measure and a proof of the fact that the model is complete if and only if there is a unique dominated martingale measure. You should also show that the minimal martingale measure minimizes $\mathbb{E}\,(d\mathbb{Q}/d\mathbb{P})^2$ over all dominated martingale measures $\mathbb{Q}$.

2 Give a short description of the standard binomial model operating over the time periods $0, 1, \ldots, n$.

Derive an expression in terms of the stock price S_n for the Radon–Nikodym derivative $d\mathbb{Q}/d\mathbb{P}$ of the martingale probability $\mathbb{Q}$ with respect to the underlying probability $\mathbb{P}$.

Consider an investor with initial wealth w_0 at time 0, who wishes to trade in this market so as to maximize the expected utility of his final wealth at time n. Calculate his optimal final wealth when his utility function is $v(x) = \gamma x^{1/\gamma}$, where $\gamma > 1$ is a given constant.

3 State Girsanov's Theorem and give a sketch of its proof.

At time 0, Company 1 announces a takeover bid for Company 2, in which it will exchange a fixed number r of its own shares for each share of Company 2 at a future time $t_0 > 0$. The directors of Company 2 believe that a fair cash price at time t_0 for each of their shares would be c and they are concerned that the stock price of Company 1 may go down between the announcement of the bid and t_0; for these reasons, they negotiate a deal in which the number of shares of Company 1 exchanged for each share of Company 2 should be

$$N = \max\left(r, \frac{c}{A}\right),$$

where $A = \left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n}$ is a geometric average of the share price $\{S_t\}$ of Company 1 at times $0 < t_n < t_{n-1} < \cdots < t_1 \leqslant t_0$.

In the context of the Black–Scholes model, calculate the value (per share of Company 2) of this deal at time 0.

4 Explain what is meant by a *self-financing* portfolio in the Black–Scholes model. Suppose that the value of a portfolio at time t is a function of the stock-price process $\{S_t,\ t \geqslant 0\}$ and is given by

$$p\left(S_t, t\right) = g\left(S_t, t\right) S_t + h\left(S_t, t\right) e^{-\rho(t_0 - t)},$$

where $g(x,t)$ and $h(x,t)$ are suitably smooth functions and ρ is the interest rate. Prove that this portfolio is self-financing on the time interval $[0, t_0]$ if and only if the equations

$$x\frac{\partial g}{\partial x} + e^{-\rho(t_0 - t)}\frac{\partial h}{\partial x} = 0, \quad \text{and}$$
$$\frac{1}{2}\sigma^2 x^2 \frac{\partial g}{\partial x} + x\frac{\partial g}{\partial t} + e^{-\rho(t_0 - t)}\frac{\partial h}{\partial t} = 0$$

are satisfied for $0 \leqslant t \leqslant t_0$, where σ is the volatility.

Deduce that the portfolio with value p is self-financing if and only if the function p satisfies the Black–Scholes equation.

Explain what changes to the Black–Scholes equation would be necessary when the stock pays a continuous dividend at the rate θS_t per unit time at time t.

5 Let $\{W_t^\nu,\ t \geqslant 0\}$ denote a standard Brownian motion with drift ν and let $M_t^\nu = \sup_{0 \leqslant s \leqslant t} W_s^\nu$. By using the Reflection Principle and Girsanov's Theorem, or otherwise, prove that for $a > 0$ and $x \leqslant a$,

$$\mathbb{P}\left(W_t^\nu \leqslant x,\, M_t^\nu < a\right) = \Phi\left(\frac{x - \nu t}{\sqrt{t}}\right) - e^{2a\nu}\Phi\left(\frac{x - 2a - \nu t}{\sqrt{t}}\right),$$

where Φ is the standard normal distribution function.

In the context of the Black–Scholes model, consider a down-and-in claim that pays $f\left(S_{t_0}\right)$ at time t_0 if a barrier $b < S_0$ is reached by the stock-price process $\{S_t,\ t \geqslant 0\}$ during the lifetime $[0, t_0]$ of the claim; otherwise it pays nothing. Show that the price at time 0 of this claim is the same as that of an ordinary terminal-value claim, paying $g\left(S_{t_0}\right)$ at t_0, where

$$g(x) = f(x) I_{(x \leqslant b)} + (1/\kappa)^{\nu/\sigma} f(x/\kappa) I_{(x > \kappa b)},$$

and ν and κ are constants which should be specified.

For a down-and-in European call option with strike price $c > b$, explain why there will be a discontinuity in the holding in stock in the replicating portfolio at the instant the barrier is reached.

6 Write an essay on one-factor models for interest rates.

END OF PAPER