UNIVERSITY OF CAMBRIDGE

M. PHIL. IN STATISTICAL SCIENCE

Friday 7 June 2002 9 to 11

ACTUARIAL STATISTICS

*Attempt **THREE** questions*

*There are **four** questions in total*

The questions carry equal weight

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A portfolio of insurance policies gives rise to N claims in a particular time period, with independent, identically distributed claims $X_1, X_2, \ldots$, independent of N.

Find the moment generating function of the total claim $S = X_1 + \ldots + X_N$ in terms of the probability generating function G_N of N and the moment generating function M_X of X_1.

Suppose that

$$\mathbb{P}(N = n) = p_n = \binom{k+n-1}{n}(1-p)^n p^k, \qquad n = 0, 1, 2, \ldots$$

where k is a positive integer and $0 < p < 1$, and suppose that the claims are exponentially distributed with mean $\frac{1}{\lambda}$. Show that S has the same distribution as the total claim amount $\tilde{S}$ when the number of claims $\tilde{N}$ has a Binomial distribution. Show that this Binomial distribution has parameters k and $1-p$, and that the associated claim sizes are exponentially distributed.

Show that

$$\mathbb{P}(S > x) = \sum_{n=1}^{\infty} p_n \sum_{j=0}^{n-1} \frac{e^{-\lambda x}(\lambda x)^j}{j!}, \qquad x > 0$$

and find a similar expression for $\mathbb{P}(\tilde{S} > x)$. Explain why the relationship between S and $\tilde{S}$ is of practical use in calculation of $\mathbb{P}(S > x)$.

2 In a classical risk model with relative safety loading $\rho > 0$, claims arrive in a Poisson process, rate $\lambda > 0$, and claims $X_1, X_2, \ldots$ are independent exponential random variables with mean μ independent of the claim arrival process. Define the adjustment coefficient R, and find R in terms of ρ and μ.

Define the probability of ruin with initial capital $u > 0$, and state the Lundberg inequality.

Let $\psi_n(u)$, $n = 1, 2, \ldots$ be the probability of ruin on or before the n^{th} claim with initial capital u. Show that the Lundberg inequality implies that $\psi_n(u) \leqslant e^{-Ru}$ for all $n = 1, 2, \ldots$ and for all $u > 0$.

If $\mu = 1$, show that $\psi_1(u) = \dfrac{e^{-u}}{2+\rho}$ and find $\psi_2(u)$.

Verify directly that $\psi_n(u) \leqslant e^{-Ru}$ for $n = 1, 2$ in this case.

3 Describe the Bühlmann model and derive the credibility factor and the credibility premium for this model.

The number of claims N in a year has a Poisson distribution with mean θ where θ has prior density

$$\pi(\theta) = \frac{3}{\theta^4}, \quad \theta > 1$$

(zero otherwise). A particular policy has a total of k claims over the previous two years. Find the Bühlmann model credibility estimate for the expected number of claims in the next year.

If $k = 5$, find the Bayesian credibility estimate, and show that it differs from the Bühlmann credibility estimate in this case.

4 A No Claims Discount System has three levels of discount, 0%, 100α%, 100β%, $0 < \alpha < \beta < 1$, and full premium c. If a policy-holder makes no claims in a year, then he moves up to the next (higher) level of discount (or stays at 100β%). If the policy-holder is at the 0% and 100α% levels, and at least one claim is made, then he moves down to (or stays at) the 0% level. If a policy-holder at the 100β% level makes exactly one claim during a year, he moves down one discount level, but if two or more claims are made, he moves down two levels.

(a) Suppose that, conditional on λ, the number of claims N in a year has a Poisson distribution with mean λ, and λ is a random variable with density $f(\lambda) = \nu e^{-\nu\lambda}$ $(\lambda > 0)$ for some $\nu > 0$. Show that $\mathbb{P}(N = k) = pq^k$ for $k = 0, 1, \ldots$ where $q = 1 - p = (1 + \nu)^{-1}$. Show that the transition matrix for this system is

$$\begin{bmatrix} q & 1-q & 0 \\ q & 0 & 1-q \\ q^2 & q(1-q) & 1-q \end{bmatrix}$$

and hence find the expected premium in the steady state.

(b) Suppose instead that the losses associated with claim events are independent random variables with density $f(x) = e^{-x}$ $(x > 0)$. A policy-holder in the highest discount category decides whether or not to claim for any losses by comparing costs and premiums over the infinite time horizon, assuming no further claim events. Find the probability that this policy-holder claims on his first loss L_1 in a given year. He then incurs a second loss L_2 in the same year. Find the probability that he claims for L_2, given that he claimed for L_1.