# Probability and Statistics 1 2011–12

# Progress Test

11 January 2012

**Answer all 6 questions.**
**The total number of marks available is 100.**

**1.** $X$ is a continuous random variable with density function

$$f(x) = c|x|(1-x^2) \quad \text{for } -1 < x < 1$$

with $f(x) = 0$ for $|x| > 1$. In this formula, $c$ is a positive constant.

(i) Sketch the density function $f$. **[3 marks]**

(ii) Evaluate the constant $c$ and hence find the expectation and variance of $X$. **[7 marks]**

(iii) For a given $0 \le x \le 1$, calculate $\mathbb{P}(-x < X \le 0)$ and $\mathbb{P}(0 < X \le x)$. **[4 marks]**

(iv) Suppose that $Y$ is defined by $Y = X^2$. Calculate the distribution function $F_Y(y)$ of $Y$ and deduce the density function $f_Y(y)$. **[6 marks]**

*[Total: 20 marks]*

**2.** A motor insurance company has sold 150 insurance policies. Let $N_i$ represent the number of claims made on policy $i$. You may assume that $N_1$, $N_2$, $\ldots$, $N_{150}$ is a sequence of independent Poisson random variables, each with mean 0.2.

(i) Evaluate the probability that at least two claims are made on a given policy. **[3 marks]**

(ii) A manager looks through the policies to find one on which two or more claims have been made. Calculate the probability that the manager has to look through at least 50 policies before finding one. **[4 marks]**

(iii) Let $Y$ denote the total number of policies on which at least two claims have been made.

a) Write down the distribution of $Y$ and state its mean and variance.

b) Calculate $\mathbb{P}(Y \ge 3)$. (You may, if you wish, use a suitable approximation but, if you do, you should provide justification that the approximation is reasonable in this case.)

**[9 marks]**

*[Total: 16 marks]*

**3.** Two independent cricket matches are played between two teams, A and B. Before each match a fair coin is tossed; the team that wins the toss has an advantage, winning the match with probability 0.4, whereas the other team has probability 0.3 of winning and there is a 30% chance that the match ends in a draw. Calculate

(i) the probability that team A wins the first match; **[3 marks]**

(ii) the probability that team A wins both matches; **[2 marks]**

(iii) the probability that team A wins more matches than team B; **[4 marks]**

(iv) the conditional probability that team A won the toss before both matches, given that team A won both matches. **[4 marks]**

*[Total: 13 marks]*

**4.** (i) a) Show the events $A \cap B$ and $A^c \cap B^c$ on a Venn diagram.

b) Suppose that $A$ and $B$ are independent events. Prove that $A^c$ and $B^c$ are independent events. **[11 marks]**

(ii) A multiple choice question has four possible answers, (a) to (d), of which two are correct. Let $A$ be the event that (c) is incorrect, $B$ the event that the two correct answers are next to each other on the question paper, and $C$ the event that (d) is correct.

a) Interpret in words the event $A^c \cap (B \cup C)$.

b) Is it true to say that $B \cap A^c \subset C$? Explain your answer.

c) Write the event "The correct answers are (a) and (b)" in terms of $A$, $B$ and $C$.

**[7 marks]**

*[Total: 18 marks]*

**5.** A Statistics exam is taken by 10 students on a Philosophy degree and 8 students on a Zoology degree. The marks they achieve are as follows:

| | | | | | | | | | | | sample mean |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Philosophy students | 25 | 32 | 38 | 44 | 44 | 48 | 50 | 58 | 62 | 92 | 49.3 |
| Zoology students | 40 | 45 | 53 | 55 | 58 | 62 | 69 | 78 | | | 57.5 |

(i) Calculate the median and quartiles of each data set and identify any outliers which may be present. **[8 marks]**

(ii) Use a box-and-whisker diagram to illustrate the two data sets. **[4 marks]**

(iii) Compare the two data sets in terms of location, spread and skewness. **[6 marks]**

*[Total: 18 marks]*

**6.** The marks scored on an exam paper are reported as follows:

| Mark range | $< 20$ | 20–39 | 40–49 | 50–59 | 60–69 | 70–79 | 80–100 |
|---|---|---|---|---|---|---|---|
| Number of scripts | 0 | 4 | 12 | 20 | 21 | 9 | 5 |

(i) Use a suitable diagram to illlustrate the data. **[5 marks]**

(ii) Calculate the sample mean and sample variance of the data. **[4 marks]**

(iii) Evaluate the median and quartiles of the data, correct to 1 decimal place. **[6 marks]**

*[Total: 15 marks]*

# Formulae

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).$$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

| Distribution | Notation | Mean | Variance | $p(x)$ | Domain |
|---|---|---|---|---|---|
| Binomial | $\text{Bin}(k, \theta)$ | $k\theta$ | $k\theta(1-\theta)$ | $\binom{k}{x}\theta^x (1-\theta)^{k-x}$ | $x = 0, 1, \ldots, k$ |
| Poisson | $\text{Pois}(\lambda)$ | $\lambda$ | $\lambda$ | $e^{-\lambda}\dfrac{\lambda^x}{x!}$ | $x = 0, 1, \ldots$ |
| Geometric | $\text{Geom}(\theta)$ | $\theta^{-1}$ | $\theta^{-2}(1-\theta)$ | $\theta (1-\theta)^{x-1}$ | $x = 1, 2, \ldots$ |