Probability and Statistics 1 2011–12

Solutions to Progress Test

1. (i) The sketch should demonstrate that the density is symmetric about 0.

(ii) $1 = -c\int_{-1}^{0} x(1-x^2)\,dx + c\int_0^1 x(1-x^2)\,dx = 2c\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{c}{2}$. Hence $c = 2$.
By symmetry, $\mathbb{E}(X) = 0$.
Therefore $\mathrm{Var}\,(X) = \mathbb{E}(X^2) = 2\int_0^1 (2x^3 - 2x^5)\,dx = 4\left(\frac{1}{4}-\frac{1}{6}\right) = \frac{1}{3}$.

(iii) $\mathbb{P}(0 < X \le x) = \int_0^x cy(1-y^2) = x^2 - \frac{1}{2}x^4$ for $0 \le x \le 1$. $\mathbb{P}(-x < X \le 0)$ is the same, by symmetry.

(iv) $\mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = 2y - y^2$ for $0 \le y \le 1$.
Therefore $f_Y(y) = 2(1-y)$ for $0 < y < 1$.

2. (i) The probability that there are two or more claims on an individual policy is $\theta = 1 - e^{-0.2}(1+0.2) = 1 - 1.2e^{-0.2} = 0.0175$.

(ii) The number of policies, X, which have to be considered before finding one with 2 or more claims has a geometric distribution with parameter θ. $\mathbb{P}(X \ge 50) = \sum_{50}^{\infty} \theta(1-\theta)^{x-1} = (1-\theta)^{49} = 0.4205$.

(iii)

a) The distribution is Binomial with parameters 150 and θ.
The expectation is $150\theta = 2.63$, the variance $150\theta(1-\theta) = 2.58$.

b) Using binomial probabilities, we get

$$\mathbb{P}(Y \ge 3) = 1 - [p_Y(0) + p_Y(1) + p_Y(2)] = 1 - [0.0705 + 0.1887 + 0.2507] = 0.4901.$$

Instead, we could use the Poisson approximation, since θ is small and $k\theta < 5$. This gives us

$$\mathbb{P}(Y \ge 3) \approx 1 - [p_Y(0) + p_Y(1) + p_Y(2)] = 1 - [0.0722 + 0.1897 + 0.2494] = 0.4887.$$

3. (i) This is $\frac{1}{2} \times 0.4 + \frac{1}{2} \times 0.3 = \frac{7}{20}$ or 0.35.

(ii) By independence, the answer is $0.35^2 = 0.1225$ or $\frac{49}{400}$.

(iii) We already have the probability of a 2–0 scoreline in the series. We need to add the probability of a 1–0 scoreline, i.e. the probability that team A wins one and the other is drawn. This is $2 \times 0.35 \times 0.3 = 0.21$. Adding this to the previous answer gives 0.3325 or $\frac{133}{400}$.

(iv) The probability that team A won the toss both times and won both matches is $(\frac{1}{2} \times 0.4)^2 = 0.04$.
Therefore the answer is $\frac{0.04}{0.1225} = 0.3265 = \frac{16}{49}$

4. (i)

a) Diagram.

b) By independence, $\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$.

$$\begin{aligned}\mathbb{P}(A^c \cap B^c) &= 1 - \mathbb{P}(A \cup B) = 1 - [\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)] \\ &= 1 - \mathbb{P}(A) - \mathbb{P}(B) + \mathbb{P}(A)\,\mathbb{P}(B) = [1 - \mathbb{P}(A)][1 - \mathbb{P}(B)] \\ &= \mathbb{P}(A^c)\mathbb{P}(B^c).\end{aligned}$$

(ii)

a) The event that (c) is correct and either the two correct answers are adjacent or (d) is also correct.

b) No. $B \cap A^c$ includes the possibility that (b) and (c) are the correct answers, which is not an element of C.

c) $A \cap B$ or $A \cap B \cap C^c$ would do here.

5. (i) Philosophy: 38, 46, 58; Zoology: 49, 56.5, 65.5.
Philosophy: IQR is 20; 92 is a possible outlier.
Zoology: IQR is 16.5; no outliers.

(ii) Diagram.

(iii) Location: The quartiles for Z are all higher than those for P, as is the sample mean.
Spread: P has a higher IQR as well as a larger range, so it is more spread out.
Skewness: P seems positively skewed, based on inter-quartile distances and on comparison of mean with median. Z is more symmetric, based on the same data, but slightly positively skewed.

6. (i) A histogram would be most appropriate, though marks would also be awarded for a correctly drawn cumulative frequency diagram.

(ii) $\bar{x} = \frac{\sum x f_x}{n} = \frac{4217}{71} = 59.39$.
$s_x^2 = \frac{\sum x^2 f_x - n\bar{x}^2}{n-1} = \frac{14000}{70} = 200$.

(iii) It is intended that linear interpolation be used here. There are 71 observations.
LQ: in the 50–59 range. We solve $\frac{x-49.5}{10} = \frac{17.75-16}{20}$, giving a LQ equal to $49.5 + 0.88 = 50.4$ to 1dp.
Median: in the 50–59 range. We solve $\frac{x-49.5}{10} = \frac{35.5-16}{20}$, giving a LQ equal to $49.5 + 9.75 = 59.3$ to 1dp.
UQ: in the 60–69 range. We solve $\frac{x-59.5}{10} = \frac{53.25-36}{21}$, giving a LQ equal to $59.5 + 8.21 = 67.7$ to 1dp.