**Probability and Statistics 1 - Surgery Hours class (Andres Villegas)**
**Exercise Sheet 8 Solutions: Continuous Random Variables 2**

1.

a) Let $X$ be the temperature on May 5. Then $X$ has a normal distribution with $\mu = 24$ and $\sigma = 3$. The desired probability is given by

$$P(X > 27) = P\left(\frac{X-24}{3} > \frac{27-24}{3}\right) = P(Z > 1)$$
$$= 1 - P(Z \le 1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

b) Let $Y$ be the number of times with broken records during the next 5 years on May 5. Then, $Y$ has a binomial distribution with $n = 5$ and $p = 0.1587$. So, the desired probability is

$$P(Y \ge 3) = P(Y = 3) + P(Y = 4) + P(Y = 5)$$
$$= \binom{5}{3}(0.1587)^3(0.8413)^2 + \binom{5}{4}(0.1587)^4(0.8413)^1 + \binom{5}{5}(0.1587)^5(0.8413)^0$$
$$\approx 0.03106$$

2. Since the standard normal density $f_Z(z)$ is symmetric around 0, we have that

$$F_Z(-z) = \int_{-\infty}^{-z} f_Z(u)du = \int_{z}^{\infty} f_Z(u)du = 1 - F_Z(z) = 1 - \alpha.$$

In addition, we have that

$$P(-z \le Z \le z) = \int_{-z}^{z} f_Z(u)du = F_Z(z) - F_Z(-z) = \alpha - (1 - \alpha) = 2\alpha - 1$$

3. Let $X$ be the actual amount put into each bottle. Then $X$ has a normal distribution with $\mu = 360$ and $\sigma = 4$.

a) The desired probability is given by

$$P(X \le 355) = P\left(\frac{X-360}{4} \le \frac{355-360}{4}\right) = P(Z \le -1.25)$$
$$= 1 - \Phi(1.25) = 1 - 0.8944 = 0.1056$$

b) We need to find $\mu$ such that with $X \sim N(\mu, \sigma^2 = 4^2)$, $P(X \le 355) = 0.025$. We have

$$P(X \le 355) = P\left(\frac{X-\mu}{4} \le \frac{355-\mu}{4}\right) = P\left(Z \le \frac{355-\mu}{4}\right)$$

We know that $P(Z < -1.96) = 0.025$, hence $-1.96 = \frac{355-\mu}{4}$. Solving for $\mu$ we find that the machine should be adjusted to $\mu = 362.84$ml.

4. Let $X$ be the number of people voting in favour of the facility being built. Then $X \sim Bin(200, 0.47)$. The probability the poll show a majority in favour is given by $P(X > 100) = 1 - P(X \le 100)$. Using a normal approximation with continuity correction we have that $P(X \le 100) \approx P(Y \le 100.5)$, where $Y \sim N(200(0.47) = 94, 200(0.47)(0.53) = 49.82)$. Thus,

$$P(Y \le 100.5) = P\left(\frac{Y-94}{\sqrt{49.82}} \le \frac{100.5-94}{\sqrt{49.82}}\right) = P(Z \le 0.92) = \Phi(0.92) = 0.8212$$

Then

$$P(X > 100) = 1 - P(X \le 100) \approx 1 - P(Y \le 100.5) = 1 - 0.8212 = 0.1788$$

5. Let $T$ be the time in hours between arrival of customers at the service window. Then $T$ has an exponential distribution with parameter $\lambda = \frac{1}{12/60} = 5$. Therefore, the number of arrivals per hour $X$ has a Poisson distribution with mean $\lambda = 5$. Hence

$$P(X = 10) = \frac{e^{-5}5^{10}}{10!} = 0.01813$$

6. For $10{,}000e^{0.04} \leq v \leq 10{,}000e^{0.08}$, the cumulative distribution function $F_V(v)$ is given by

$$F_V(v) = P(V \leq v) = P(10000e^R \leq v) = P\left(R \leq \log\left(\frac{y}{10{,}000}\right)\right) = F_R\left(\log\left(\frac{y}{10{,}000}\right)\right)$$

Since $R$ has a uniform distribution on $(0.04, 0.08)$, $F_R(r) = \frac{r-0.04}{0.04}$ for $0.04 \leq r \leq 0.08$. Hence

$$F_V(v) = F_R\left(\log\left(\frac{y}{10{,}000}\right)\right) = \frac{\log\left(\frac{y}{10{,}000}\right) - 0.04}{0.04}$$

In addition for $v < 10{,}000e^{0.04}$ $F_V(v) = 0$ and for $v > 10{,}000e^{0.08}$ $F_V(v) = 1$.

7. We have that

$$F_Y(y) = P(Y \leq y) = P(10X^{0.8} \leq y) = P\left(X \leq \left(\frac{y}{10}\right)^{1.25}\right) = F_X\left(\left(\frac{y}{10}\right)^{1.25}\right)$$

Since $X$ has an exponential distribution with mean 1, $F_X(x) = 1 - e^{-x}$ . Hence

$$F_Y(y) = F_X\left(\left(\frac{y}{10}\right)^{1.25}\right) = 1 - e^{-\left(\frac{y}{10}\right)^{1.25}}$$

Therefore, the density function of $Y$ is given by

$$f_Y(y) = F'_Y(y) = 0.125\left(\frac{y}{10}\right)^{0.25} e^{-\left(\frac{y}{10}\right)^{1.25}}$$