Probability and Statistics 1 - Surgery Hours class (Andres Villegas)
Exercise Sheet 7 Solutions: Continuous Random Variables 1

1.

a) We have that

$$\int_0^1 ax^2 = 1$$
$$a\frac{x^3}{3}\bigg|_0^1 = 1$$
$$\frac{a}{3} = 1$$

Hence $a = 3$.

b) We have

$$\begin{aligned} P\left(Y > \frac{1}{2}\right) &= P\left(\max\{X_1, X_2, X_3\} > \frac{1}{2}\right) \\ &= 1 - P\left(\max\{X_1, X_2, X_3\} \le \frac{1}{2}\right) \\ &= 1 - P\left(X_1 \le \frac{1}{2}, X_2 \le \frac{1}{2}, X_3 \le \frac{1}{2}\right) \end{aligned}$$

But, since X_1, X_2, X_3 are independent

$$P\left(X_1 \le \frac{1}{2}, X_2 \le \frac{1}{2}, X_3 \le \frac{1}{2}\right) = P\left(X_1 \le \frac{1}{2}\right) P\left(X_2 \le \frac{1}{2}\right) P\left(X_3 \le \frac{1}{2}\right)$$

In addition

$$P\left(X_i \le \frac{1}{2}\right) = \int_0^{1/2} 3x^2 = 3\frac{x^3}{3}\bigg|_0^{1/2} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Therefore,

$$\begin{aligned} P\left(Y > \frac{1}{2}\right) &= 1 - P\left(X_1 \le \frac{1}{2}, X_2 \le \frac{1}{2}, X_3 \le \frac{1}{2}\right) \\ &= 1 - P\left(X_1 \le \frac{1}{2}\right) P\left(X_2 \le \frac{1}{2}\right) P\left(X_3 \le \frac{1}{2}\right) \\ &= 1 - \left(\frac{1}{8}\right)^3 \\ &= 1 - \frac{1}{512} \\ &= \frac{511}{512} \end{aligned}$$

2.

a) $f(x)$ must satisfy

$$\int_{1/2}^{3/2} kx(c2 - x)dx = 1$$

$$k\left(\frac{c}{2}x^2 - \frac{2}{3}x^3\right)\Big|_{\frac{1}{2}}^{\frac{3}{2}} = 1$$

$$k\left[\frac{c}{2}\left(\frac{3}{2}\right) - \frac{2}{3}\left(\frac{3}{2}\right)^3 - \frac{c}{2}\left(\frac{1}{2}\right)^2 + \frac{2}{3}\left(\frac{1}{2}\right)^3\right] = 1$$

$$k\left(c - \frac{13}{6}\right) = 1$$

In addition we know that

$$E(X) = 0.9$$

$$\int_{\frac{1}{2}}^{\frac{3}{2}} xf(x) = 0.9$$

$$\int_{\frac{1}{2}}^{\frac{3}{2}} kx^2(c - 2x) = 0.9$$

$$k\left(\frac{c}{3}x^3 - \frac{1}{2}x^4\right)\Big|_{\frac{1}{2}}^{\frac{3}{2}} = 0.9$$

$$k\left[\frac{c}{3}\left(\frac{3}{2}\right)^3 - \frac{1}{2}\left(\frac{3}{2}\right)^4 - \frac{c}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{1}\left(\frac{1}{2}\right)^4\right] = 0.9$$

$$k\left(\frac{13}{12}c - \frac{5}{2}\right) = 0.9$$

Therefore we know that k and c satisfy $k\left(c - \frac{13}{6}\right) = 1$ and $k\left(\frac{13}{12}c - \frac{5}{2}\right) = 0.9$. Dividing these two equations we get

$$\frac{\frac{13}{12}c - \frac{5}{2}}{c - \frac{13}{6}} = \frac{9}{10}$$

$$\frac{13}{12}c - \frac{5}{2} = \frac{9}{10}c - \frac{117}{60}$$

$$\left(\frac{13}{12} - \frac{9}{10}\right)c = \frac{5}{2} - \frac{117}{60}$$

$$\frac{11}{60}c = \frac{11}{20}$$

$$c = 3$$

And therefore

$$k = \frac{1}{c - \frac{13}{6}} = \frac{1}{3 - \frac{13}{6}} = \frac{6}{5}$$

b) Let Y be the number of gallons sold. We have

$$Y = \begin{cases} 1000X & if \quad X \leq 1 \\ 1000 & if \quad X > 1 \end{cases}$$

Then

$$E(Y) = \int_{\frac{1}{2}}^{1} 1000xf(x)dx + \int_{1}^{\frac{3}{2}} 1000f(x)dx$$
$$= \int_{\frac{1}{2}}^{1} 1000\left(\frac{6}{5}\right)x^2(3-2x)dx + \int_{1}^{\frac{3}{2}} 1000\left(\frac{6}{5}\right)x(3-2x)dx$$
$$= 1200\left(\int_{\frac{1}{2}}^{1} x^2(3-2x)dx + \int_{1}^{\frac{3}{2}} x(3-2x)dx\right)$$
$$= 1200\left(\left(x^3 - \frac{1}{2}x^4\right)\Big|_{\frac{1}{2}}^{1} + \left(\frac{3}{2}x^2 - \frac{2}{3}x^3\right)\Big|_{1}^{\frac{3}{2}}\right)$$
$$= 1200(0.40632 + 0.291667)$$
$$E(Y) = 837.5$$

3.

a) $E(T) = \frac{100+0}{2} = 50, V(T) = \frac{(100-0)^2}{12} = 833.3333$

b) In general $P(T > t) = \int_t^{100} \frac{1}{100} dx = \frac{100-t}{100}$. Therefore $P(T > 57) = \frac{100-57}{100} = 0.43$

c) For $40 \leq t \leq 100$, we have that

$$P(S > t) = P(T > t | T > 40)$$
$$= \frac{P(T > t, T > 40)}{P(T > 40)}$$
$$= \frac{P(T > t)}{P(T > 40)}$$
$$= \frac{\frac{100-t}{100}}{\frac{100-40}{100}}$$
$$= \frac{100-t}{60},$$

It follows then that $F_S(t) = P(S \leq t) = 1 - P(S > t) = 1 - \frac{100-t}{60}$, for $40 \leq t < 100$, which corresponds to the cumulative distribution function of a uniform random variable over [40,100].

d) $E(S) = \frac{100+40}{2} = 70$ and $P(S > 57) = \frac{100-57}{60} = 0.7167$.

4. Let T be the time of failure of the printer. The expected cost of refunds is given by

$$E(cost\ of\ refunds) = 100(200P(T \leq 1) + 100P(1 < T \leq 2))$$

Since $T \sim Exp(0.5)$ we have that $F(t) = P(T \leq t) = 1 - e^{-0.5t}$. Hence

$$E(cost\ of\ refunds) = 100[200P(T \leq 1) + 100P(1 < T \leq 2)]$$
$$= 100\left[200F(1) + 100\left(F(2) - F(1)\right)\right]$$
$$= 100\left[200(1 - e^{-0.5}) + 100\left(e^{-0.5} - e^{-0.5(2)}\right)\right]$$
$$= 100[200(0.3937) + 100(0.2386)]$$
$$= 10255.90$$

5. By the law of total probability

$$\begin{aligned} P(4 < S < 8\,) &= P(4 < S < 8 \mid N = 0)P(N = 0) + P(4 < S < 8 \mid N = 1)P(N = 1) \\ &\quad + P(4 < S < 8 \mid N > 1)P(N > 1) \end{aligned}$$

We have

$$P(4 < S < 8 \mid N = 0) = 0$$

$$P(4 < S < 8 \mid N = 1) = \int_4^8 \frac{1}{5} e^{-\frac{1}{5}x} dx = e^{-\frac{4}{5}} - e^{-\frac{8}{5}} = 0.2474$$

$$P(4 < S < 8 \mid N > 1) = \int_4^8 \frac{1}{8} e^{-\frac{1}{8}x} dx = e^{-\frac{4}{8}} - e^{-\frac{8}{8}} = 0.2387$$

Hence

$$P(4 < S < 8\,) = 0\left(\frac{1}{2}\right) + 0.2474\left(\frac{1}{3}\right) + 0.2387\left(\frac{1}{6}\right) = 0.1223$$