Probability and Statistics 1 - Surgery Hours class (Andres Villegas)
Exercise Sheet 5 Solutions: Discrete Random Variables 1

1. Let Y be the random variable for the number in the selected ball. We have that Y has a discrete uniform distribution over $\{1,2,\ldots,25\}$. Hence, $E(Y)=\frac{1}{2}(25+1)=13$ and $Var(X)=\frac{1}{12}(25-1)(25-1+2)=52$. Since $X=1000Y$, then $E(X)=E(1000Y)=1000E(X)=1000(13)=13000$ and $Var(X)=Var(1000Y)=1000^2Var(X)=1000^2(52)=52000000$. Thus, the mean of X is £13000 and the standard deviation is £√52000000=£7211.10

2.
   a) Using C for a claim, N for no claim, then
$$P(premium=400)=P(C\ in\ year\ 3, regardless\ of\ the\ first\ 2\ years)=p$$
$$P(premium=400k)=P\left(CN\ in\ year\frac{2}{3}, regardless\ of\ the\ first\ year\right)=p(1-p)$$
$$P(premium=400k^2)=P\left(NN\ in\ year\frac{2}{3}, regardless\ of\ the\ first\ year\right)=(1-p)^2$$
   b)
$$E(premium)=400p+400kp(1-p)+400k^2(1-p)^2$$
$$=400\{p+kp(1-p)+k^2(1-p)^2\}$$
   c) For $E(premium)=300$ when $p=0.1$ then
$$400\{0.1+(0.1)(009)k+0.9^2k^2\}=300$$
$$0.1+0.09k+0.81k^2=0.75$$
$$0.81k^2+0.09k-0.65=0$$
   Hence
$$k=\frac{-0.09\pm\sqrt{0.09^2+4(0.81)(0.65)}}{2(0.81)}=\frac{-0.09+1.454}{1.62}=0.84$$

3.
   a) We have that
$$E(Revenue)=E(income\ from\ tickets)-E(expenditure\ for\ overbooking)$$
   Clearly $E(income\ from\ tickets)=21(50)=1050$.
   Let N be the random variable for the number of tourists that show up. We have that $Y\sim Binomial(21,0.98)$. Thuse, the probability of having to pay a penalty due to overbooking is $P(Y=21)=\binom{21}{21}0.98^{21}=0.98^{21}=0.6543$. Therefore $E(expenditure\ for\ overbooking)=100P(Y=21)=100(0.6543)=65.43$, and the expected revenue of the tour operator is 1050-65.43=984.57.
   b) If the tour operator does not overbook (i.e. sells only 20 tickets) the expected revenue would be $20(50)=1000$ which is greater than the expected revenue under the overbooking strategy. Therefore, overbooking is not a profitable strategy for the tour operator.
   c) For overbooking to be profitable it must hold that
$$E[Revenue]\geq 1000$$
$$1050-100(1-p)^{21}\geq 1000$$
$$(1-p)^{21}\leq 0.5$$

$$p \geq 1 - 0.5^{\frac{1}{21}}$$
$$p \geq 0.0325$$

4. Let N_1 and N_2 be the number of participants that complete the study in groups 1 and 2, respectively. We have that $N_i \sim Binomial(10{,}0.8)$. Therefore, for each group, the probability that at least 9 participants complete the study is given by

$$P(N_i \geq 9) = P(N_i = 9) + P(N_i = 10)$$
$$= \binom{10}{9}(0.8)^9(0.2) + \binom{10}{10}(0.8)^{10}$$
$$= 10(0.8)^9(0.2) + (0.8)^{10}$$
$$= 0.2684 + 0.1074 = 0.3758$$

The required probability is

$$\begin{aligned} P(N_1 \geq 9, N_2 < 9 \; or \; N_2 \geq 9, N_1 < 9) &= P(N_1 \geq 9, N_2 < 9) + P(N_2 \geq 9, N_1 < 9) \\ &= P(N_1 \geq 9)P(N_2 < 9) + P(N_2 \geq 9)P(N_1 < 9) \\ &= 0.3758(1 - 0.3758) + 0.3758(1 - 0.3758) \\ &= 0.4691 \end{aligned}$$

5. Let X be the event that the shipment comes from company X and let N be the random variable for the number of ineffective vials in the random sample. Using this notation the required probability is $P(X|N = 1)$. By the Bayes' Theorem it follows that

$$P(X|N = 1) = \frac{P(N = 1|X)P(X)}{P(N = 1|X)P(X) + P(N = 1|{\sim}X)P({\sim}X)}$$

But

$$P(N = 1|X) = \binom{30}{1}(0.1)(0.9)^{29} = 30(0.1)(0.9)^{29} = 0.1413$$
$$P(N = 1|{\sim}X) = \binom{30}{1}(0.02)(0.98)^{29} = 30(0.02)(0.98)^{29} = 0.3340$$

Hence

$$P(X|N = 1) = \frac{P(N = 1|X)P(X)}{P(N = 1|X)P(X) + P(N = 1|{\sim}X)P({\sim}X)}$$
$$= \frac{(0.1413)\left(\frac{1}{5}\right)}{(0.1413)\left(\frac{1}{5}\right) + (0.3340)\left(\frac{4}{5}\right)} = 0.0957$$

6. Let N be the number of employees that achieve a high performance. We need to find C such $P(CN > 120) < 0.01$ which is equivalent to $P\left(N \leq \frac{120}{C}\right) > 0.99$. The random variable N has a binomial distribution with parameters $n = 20$ and $p = 0.02$. Therefore we can tabulate the probability function and the distribution function of N.

x	$p(x)$	$F(x) = P(N \leq x)$
0	$\binom{20}{0}(0.98)^{20} = 0.6676$	0.6675
1	$\binom{20}{1}(0.02)(0.98)^{19} = 0.2725$	0.9401
2	$\binom{20}{2}(0.02)^2(0.98)^{18} = 0.0528$	0.9929
3	$\binom{20}{3}(0.02)^3(0.98)^{17} = 0.0065$	0.9994

Since the smallest x for which $P(N \leq x) > 0.99$ is $x = 2$, it must hold that $\frac{120}{C} \geq 2$ to satisfy $P\left(N \leq \frac{120}{C}\right) > 0.99$, and hence the maximum value of C for which $P\left(N \leq \frac{120}{C}\right) > 0.99$ is $C = \frac{120}{2} = 60.$