Probability and Statistics 1 - Surgery Hours class (Andres Villegas)
Exercise Sheet 3 Solutions: Axioms of probability, independence and conditional probability

1. a)

$$\begin{aligned} P(B^c \cap C^c) &= P((B \cup C)^c) = 1 - P(B \cup C) \\ &= 1 - \big(P(B) + P(C) - P(B \cap C)\big) \\ &= 1 - P(B) - P(C) + P(B \cap C) \\ &= 1 - P(B) - P(C) + P(B)P(C) \\ &= 1 - 0.6 - 0.4 + (0.6)(0.4) \\ &= 0.24 \end{aligned}$$

b)

$$\begin{aligned} P(B^c \cap C^c) &= 1 - P(B) - P(C) + P(B \cap C) \\ &= 1 - 0.6 - 0.4 + 0 \\ &= 0 \end{aligned}$$

c) Since $B \cap C \subset B, C$ it follows that $P(B \cap C) \leq \min\{\{P(B), P(C)\}$. Therefore

$$\begin{aligned} P(B^c \cap C^c) &= 1 - P(B) - P(C) + P(B \cap C) \\ &\leq 1 - P(B) - P(C) + \min\{\{P(B), P(C)\} \\ &= 1 - 0.6 - 0.4 + 0.4 \\ &= 0.4 \end{aligned}$$

Note that the maximum probability is achieved in the case $C \subset B$.

2.

a) We have that $A = (A \cap B) \cup (A \cap B^c)$. Since $A \cap B$ and $A \cap B^c$ are mutually exclusive, then

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

Hence

$$P(A \setminus B) = P(A \cap B^c) = P(A) - P(A \cap B)$$

b) We have that $A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$, with $A \setminus B$, $A \cap B$, and $B \setminus A$ mutually exclusive events. Thus

$$P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$$

From exercise 2. a) $P(A \setminus B) = P(A) - P(A \cap B)$ and $P(B \setminus A) = P(B) - P(A \cap B)$, then

$$\begin{aligned} P(A \cup B) &= P(A \setminus B) + P(A \cap B) + P(B \setminus A) \\ &= P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

3.

a) It is sufficient to show that $P(A \cap B^c) = P(A)P(B^c)$. But the right hand side of the above equation is equal to

$$P(A)P(B^c) = P(A)\big(1 - P(B)\big) = P(A) - P(A)P(B)$$

Since A and B are independent it follows that $P(A \cap B) = P(A)P(B)$. Then

$$P(A)P(B^c) = P(A) - P(A)P(B) = P(A) - P(A \cap B)$$

From exercise 2. a) we have $P(A) - P(A \cap B) = P(A \cap B^c)$. Therefore

$$P(A)P(B^c) = P(A) - P(A)P(B) = P(A) - P(A \cap B) = P(A \cap B^c)$$

completing the proof.

b) From De Morgan's law

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)$$

Since A and B are independent, then $P(A \cap B) = P(A)P(B)$. Therefore, we have

$$P(A^c \cap B^c) = 1 - P(A) - P(B) + P(A \cap B) = 1 - P(A) - P(B) + P(A)P(B) \qquad (1)$$

On the other hand

$$P(A^c)P(B^c) = (1 - P(A))(1 - P(B)) = 1 - P(A) - P(B) + P(A)P(B) \qquad (2)$$

From (1) and (2) $P(A^c \cap B^c) = P(A^c)P(B^c)$, and hence A^c and B^c are independent.

4. Let A denote the event that the number 6 eventually comes up. We have

$$P(A) = 1 - P(A^c)$$

where A^c is the event that a 6 never comes up. Let A_n^c denote the event that the number 6 has not come up after n tosses of the dice. We have that $P(A_n^c) = \left(\frac{5}{6}\right)^n$, so the probability that a 6 never comes up is

$$P(A_c) = \lim_{n \to \infty} P(A_n^c) = \lim_{n \to \infty} \left(\frac{5}{6}\right)^n = 0$$

Hence $P(A) = 1 - P(A^c) = 1 - 0 = 1$, which means that that a 6 will eventually come up.

5. Let N, L, H denote the event that the policyholder files no claims, a low claim, and a high claim, respectively. We need to find

$$P(L|L \cup H) = \frac{P\big(L \cap (L \cup H)\big)}{P(L \cup H)} = \frac{P(L)}{P(L \cup H)} = \frac{P(L)}{P(L) + P(H)} = \frac{16.9\%}{16.6\% + 3.9\%} = 0.8125$$

6. We have

$$p_1 = \frac{1}{5} p_0$$

$$p_2 = \frac{1}{5} p_1 = \frac{1}{5}\left(\frac{1}{5} p_0\right) = \left(\frac{1}{5}\right)^2 p_0$$

In general $p_n = \left(\frac{1}{5}\right)^n p_0$. For the probabilities to be well defined they have to satisfy

$$\sum_{n=0}^{\infty} p_n = 1$$

Therefore,

$$\sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n p_0 = \frac{p_0}{1-\frac{1}{5}} = \frac{5}{4} p_0$$

Hence $p_0 = 4/5$. Let N be the random variable representing the number of claims that a policyholder files. The required probability is $P(N > 1)$. But

$$P(N > 1) = 1 - P(N \leq 1) = 1 - P(N = 0) - P(N = 1)$$
$$= 1 - \frac{4}{5} + \frac{1}{5}\left(\frac{4}{5}\right) = \frac{1}{25} = 0.04$$

7. Let $M, F,$ and S denote the events being male, being female, and smoking, respectively

a) $P(M \cap S) = \frac{19}{100} = 0.19$
b) $P(M) = \frac{60}{100} = 0.6$
c) $P(S) = \frac{31}{100} = 0.31$
d) $P(S|M) = \frac{P(M \cap S)}{P(M)} = \frac{19/100}{60/100} = \frac{19}{60} = 0.3167$
e) $P(M|S) = \frac{P(M \cap S)}{P(S)} = \frac{19/100}{31/100} = \frac{19}{31} = 0.6129$