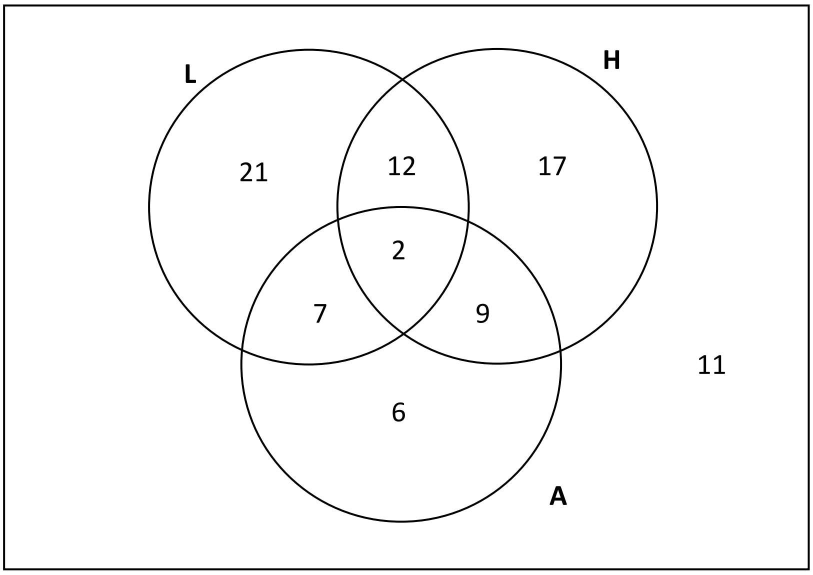
Probability and Statistics 1 - Surgery Hours class (Andres Villegas)
Exercise Sheet 2: Sets and basic probability

1. The corresponding Venn diagram is given below, where L, H, and A denote the set of people who purchased life insurance, health insurance, and auto insurance, respectively
   a) 11 purchased no policies
   b) 17 purchased only health insurance
   c) $21 + 17 + 6 = 44$ purchased exactly one type of insurance
   d) $21 + 12 + 17 = 50$ purchased life or health but not auto insurance



2. a)

$$B \setminus A = \{x: x \in B, x \notin A\} = \{x: x \in B, x \in A^c\} = B \cap A^c$$

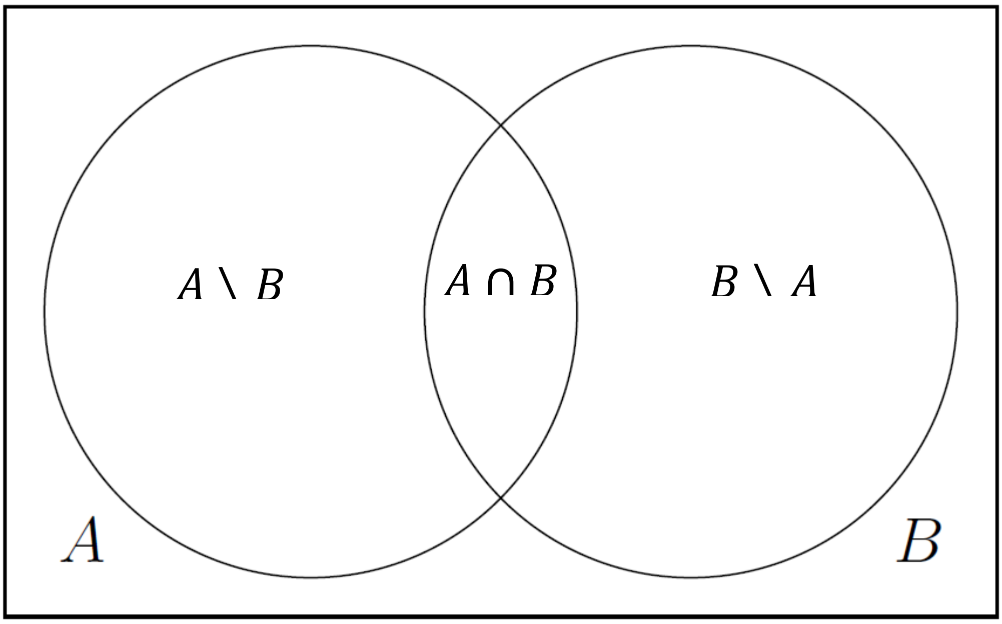
   This means that the set operation of difference can be written in terms of the operations of intersection and complementation.

   b) By the definition of union and intersection,

$$\begin{aligned} A \cap (B \cup C) &= \{x: x \in A, x \in B \cup C\} \\ &= \{x: x \in A, x \in B \quad or \quad x \in A, x \in C\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$

   Here we use the analogous logical law $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ where $\wedge$ denotes "and" and $\vee$ denotes "or".

3. The Venn diagram below illustrates the statements in this exercise.



Let U denote the universal set.

a) In order to prove that A is the disjoint union of $A \setminus B$ and $A \cap B$ we have to prove that

i) $(A \setminus B) \cup (A \cap B) = A$ (Union)

ii) $(A \setminus B) \cap (A \cap B) = \emptyset$. (Disjoint)

To show i) we have

$$(A \setminus B) \cup (A \cap B) = (A \cap B^c) \cup (A \cap B) = A \cap (B^c \cup B) = A \cap U = A$$

To show ii) we have

$$(A \setminus B) \cap (A \cap B) = (A \cap B^c) \cap (A \cap B) = A \cap (B^c \cap B) = A \cap \emptyset = \emptyset$$

b) In order to prove that $A \cup B$ is the disjoint union of $A \setminus B$ and $A \cap B$, and $B \setminus A$ we have to prove that

i) $(A \setminus B) \cup (A \cap B) \cup (B \setminus A) = A \cup B$

ii) $(A \setminus B) \cap (A \cap B) = \emptyset$, $(A \setminus B) \cap (B \setminus A) = \emptyset$, and $(B \setminus A) \cap (A \cap B) = \emptyset$

We have

$$(A \setminus B) \cup (A \cap B) \cup (B \setminus A) = A \cup (B \setminus A) = A \cup (B \cap A^C) = (A \cup B) \cap (A \cup A^c)$$

$$= (A \cup B) \cap U = A \cup B$$

From exercise 3 a) we have $(A \setminus B) \cap (A \cap B) = \emptyset$ and $(B \setminus A) \cap (A \cap B) = \emptyset$. In addition

$$(A \setminus B) \cap (B \setminus A) = (A \cap B^c) \cap (B \cap A^c) = A \cap B^c \cap B \cap A^c = \emptyset$$

completing the proof.

4. **Proof 1**: For each subset A of X, define the sequence $(a_1, a_2, \dots, a_n)$ such that $a_i = 1$ if $x_i \in A$ and 0 otherwise (For example for the empty set $\emptyset$ the associated sequence is $(0,0,\dots,0)$ and for X the associated sequence is $(1,1,\dots,1)$). Counting the number of subsets of X is equivalent to counting the possible number of sequences $(a_1, a_2, \dots, a_n)$. Since there are 2 possible values that each a_i can take, the total number of sequences is

$$\underbrace{2 \times 2 \times \dots \times 2}_{n\ times} = 2^n$$

Proof 2: We can see that enumerating the number of subsets of X is equivalent to counting all the ways of selecting k out of the n elements of X with $k = 0,1,\dots,n$. Therefore the total number of subsets is

$$\sum_{k=0}^{n} \binom{n}{k}$$

But from the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

It follows that

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

5. The probability of having no boys in the family is the probability of having three girls in the family. Since the probability of giving birth to a girl is $\frac{1}{2}$, hence the probability of having three girls is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

6. Let S denote the set of points inside the triangle, and let A denote the set of points at least one inch from any corner. S and A are pictured in the Figure below. Note that S is an equilateral triangle of side 3, and that the area outside of A is equal to half the area of a circle of radius 1. Thus

$$p = \frac{Area\ of\ A}{Area\ of\ S} = \frac{\frac{\sqrt{3}}{4}3^2 - \frac{1}{2}\pi(1)^2}{\frac{\sqrt{3}}{4}3^2} = 1 - \frac{2\pi}{9\sqrt{3}} = 1 - \frac{2\pi\sqrt{3}}{27}$$